Contents

Introduction

Let F be a non-archimedean local field and let G be a quasi-split reductive group over F. The local Langlands conjecture states that there should be a bijection between the set of equivalence classes of L-parameters $\varphi : W_F \to {}^LG$ and the set of L-packets $\Pi_\varphi(G)$, where an L-packet is a finite set of irreducible representations of $G(F)$, the L-packets are all disjoint, and their union exhausts the set of equivalence classes of irreducible smooth representations of $G(F)$.

The existence of such a bijection is known for many groups. For example, the case in which $G = T$ is a torus was already understood 60 years ago and was one of the motivating examples for this theory.

One of the most interesting cases is when we restrict our attention to the set of unramified representations of $G(F)$. We can define a map

$$\mathrm{LL}_{\mathrm{ur}} : \Pi_{\mathrm{ur}}(G) \to \Phi_{\mathrm{ur}}(G)$$

from the set of irreducible unramified representations of $G(F)$ to the set of the unramified L-parameters of G. This map is defined by taking an unramified irreducible representation π of $G(F)$, seeing it as an unramified irreducible representation of some torus $T(F) \subset G(F)$ using the Satake transform $\mathscr{S}$, and then using the correspondence for tori to get an L-parameter. The fibers of this map are almost always well understood thanks to works like [Key87] or [Mis13].

Recently, in [Kal22], the refined local Langlands conjecture was extended to some class of disconnected groups whose identity component is reductive. The goal of this thesis is to define and study unramified representations in this new disconnected setting introduced by Kaletha. In particular, we defined a map

$$\widetilde{\mathrm{LL}}_{\mathrm{ur}} : \Pi_{\mathrm{ur}}(\widetilde{G}) \to \Phi_{\mathrm{ur}}(\widetilde{G})$$

from the set of unramified representation of $\widetilde{G}(F)$ to the set of unramified L-parameters of $\widetilde{G}$, and we proved that the fibers of this map can be described as Kaletha predicted in some cases, as when the identity component $\widetilde{G}^\circ$ is GL_n or adjoint.

The structure of this thesis is as follows: in Chapter 1 we give a brief introduction to the classical local Langlands conjecture. In particular, in Section 1.3 we recall what happens in the case of unramified representations and we recall the description of the L-packets following [Key87] and [Mis13]. In Chapter 2, we follow [Kal22] for an introduction to the new disconnected conjecture. In particular, in Section 2.1 we give an easy proof of the conjecture in the case of quasi-split tori. Finally, in Chapter 3 we define and study the unramified correspondence in the disconnected setting, proving the conjecture in the case of groups with adjoint identity component. In the last section, we study the case in which there are multiple conjugacy classes of hyperspecial maximal compact subgroups, and we prove the conjecture under some strong conditions.

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Notations

Unless otherwise stated, F is a non-archimedean local field with ring of integers $\mathcal{O}_F$ and residue field $\mathfrak{f}$. We denote by q the cardinality of $\mathfrak{f}$. If F'/F is a Galois extension, we will denote by $\Gamma_{F'/F}$ the Galois group $\mathrm{Gal}(F'/F)$. We will fix $\overline{F}$ a separable closure of F, and we will denote by Γ_F the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$, by I_F the inertia subgroup, and by W_F the Weil group of F. We denote by $\mathrm{frob} \in \Gamma_F$ a Frobenius element. We denote by $\mathrm{WD_F}$ the Weil-Deligne group of F, i.e. the group scheme over $\mathbb{Q}$ given by $\mathrm{WD_F} = W_F \times \mathrm{SL}_2$.

G is a quasi-split reductive group over F with a maximal split torus S. We denote by $R(G, S)$ the set of roots associated to S, by $W = W(G, S)$ the relative Weyl group of G, and by $\widetilde{W}$ the affine Weyl group of G. We denote by $\mathcal{B}(G)$ the reduced Bruhat-Tits building of G. The apartment of the building associated to S is denoted by $\mathcal{A}(S)$. If T is any torus over F, we denote the characters (resp. the cocharacters) of T by $X^*(T)$ (resp. $X_*(T)$). $\mathcal{A}(S)$ is an affine space over $X_*(S) \otimes \mathbb{R}$, and a point $x \in \mathcal{A}(S)$ is given by a valuation of the root datum $\varphi_x = \{\varphi_{\alpha,x}\}_{\alpha \in R(G,S)}$.

1 The local Langlands conjecture for connected groups

In this section, we will introduce the unramified local Langlands conjecture for connected quasi-split reductive groups. We start by giving a fast introduction to the general conjecture using facts from [Bor79], [Car79] and [KT]. In section 1.3 we will describe the state of the art of the unramified conjecture following the original paper [Key87] and a more recent one [Mis13].

1.1 The basic objects

The automorphic side

By a representation of $G(F)$ we will mean, as customary, a pair (π, V) where V is a complex vector space and π a homomorphism $G(F) \to \mathrm{GL}(V)$. If H is a subgroup of $G(F)$, we denote by V^H the stabilizer of H in V.

Definition 1.1. A representation (π, V) of $G(F)$ is called **smooth** if the stabilizer of every vector in V is open. Equivalently, (π, V) is smooth if $V = \bigcup_K V^K$ where K runs over the compact open subgroups of $G(F)$. A smooth representation is moreover called **admissible** if V^K is finite-dimensional for every compact open subgroup K of $G(F)$. It is a non-trivial fact that every irreducible smooth representation is admissible.

Finally, (π, V) is called **tempered** if for every parabolic subgroup $P = MN$ of G, the absolute value of any character of the maximal split torus A_M in M occurring in $r_P^G \pi$ is a linear combination with non-negative coefficients of the simple roots of A_M in N. Here r_P^G is the normalized Jaquet functor [KT][3.2]. We denote by $\Pi(G)$ the set of irreducible smooth representations of $G(F)$ and by $\Pi_{\mathrm{temp}}(G)$ the set of irreducible smooth tempered representations of $G(F)$. Notice that all these definitions make sense even if G is not reductive.

The condition of being smooth imposes strong continuity conditions. Indeed, a representation (π, V) of $G(F)$ is smooth if and only if the map

$$G(F) \times V \to V$$

given by the action of $G(F)$ on V is continuous after equipping V with the discrete topology.

Every smooth representation is a $\mathbb{C}[G(F)]$-module, but not every $\mathbb{C}[G(F)]$-module is smooth (e.g. $\mathbb{C}[G(F)]$ is not smooth). For this reason we introduce the Hecke algebra of G: let K be any compact open subgroup of $G(F)$. We define $\mathcal{H}(G(F), K)$ to be the space of complex-valued functions f on $G(F)$ such that the following two conditions hold:

- f is K-bi-invariant, i.e. $f(kgk') = f(g)$ for every $k, k' \in K$ and $g \in G(F)$;
- f vanishes outside a finite union of double cosets KgK.

Choosing a left-invariant Haar measure μ on $G(F)$, we can give $\mathcal{H}(G(F), K)$ an associative algebra structure, with product given by convolution:

$$(f * f')(g) = \int_{G(F)} f(x) f'(x^{-1}g) dx.$$

A basis of this algebra is given by the set of characteristic functions on a double coset on K.

We now define $\mathcal{H}(G(F)) := \bigcup_K \mathcal{H}(G(F), K)$ where the union runs over all the possible compact open subgroups K of $G(F)$. We get that $\mathcal{H}(G(F))$ is the algebra of locally constant and compactly supported complex-valued functions on $G(F)$, with product given by convolution, and it is called the **Hecke algebra** of $G(F)$.

The Hecke algebra is the smooth analogue of the group algebra:

Theorem 1.2. *The category of non-degenerate $\mathcal{H}(G(F))$-modules is equivalent to the category of smooth representations of $G(F)$ and intertwining maps.*

Proof. The proof of this can be found in [Car79, Section 1.4]. □

Remark 1.3. From now on, when we talk about representations of reductive groups we will consider only smooth representations.

To conclude this section about smooth representations, we introduce the notion of parabolic induction of a representation: let $P \subset G$ a parabolic subgroup, let N be the unipotent radical of P and let $M \cong P/N$ be its reductive quotient. Let (π, V) be a smooth representation of $M(F)$. We will view π as its inflation to $P(F)$. The **normalized parabolic induced representation** $i_P^G \pi$ is the space of locally constant functions $f : G(F) \to V$ such that

$$f(pg) = \delta(m)^{1/2} \pi(m) f(g) \ \ \forall p = mn \in P(F), g \in G(F),$$

with left action of $G(F)$ given by $(g \cdot f)(x) = f(xg)$. Here $\delta(m) := |\det(\mathrm{Ad}_{\mathfrak{n}}(m))|_F$ is the modulus character of the action of $M(F)$ on $N(F)$, where $\mathrm{Ad}_{\mathfrak{n}}(m)$ is the adjoint action of $m \in M(F)$ on the Lie algebra $\mathfrak{n}$ of $N(F)$.. We recall that if π is admissible (resp. has finite length) then $i_P^G \pi$ is again admissible (resp. has finite length). Moreover, if $P = B$ is a Borel subgroup and $M = T$ is a maximal torus, then i_B^G takes unitary characters to unitary representations.

The Galois side

Definition 1.4. Let T be a maximal torus of G, let B be a Borel subgroup containing T, $R(G,T)$ the set of roots associated with T, and $\psi = (X^*(T), \Delta(B), X_*(T), \Delta^\vee(B))$ the based root datum associated to (G,B,T). We define the **complex dual group** $\widehat{G}$ to be the reductive group over $\mathbb{C}$ corresponding to the based root datum $\psi^\vee = (X_*(T), \Delta^\vee(B), X^*(T), \Delta(B))$. We denote by $\widehat{B}$ and $\widehat{T}$ the Borel and the maximal split torus defined by the root datum.

Example 1.5. If $G = \mathrm{GL}_n$ then $\widehat{G} \cong \mathrm{GL}_n / \mathbb{C}$: let T be the maximal torus consisting of diagonal matrices in GL_n . Let $\{\varepsilon_i\}_{i=1,\dots n}$ be the canonical basis of $X^*(T) \cong \mathbb{Z}^n$, and let $\{\varepsilon_i^\vee\}_{i=1,\dots,n}$ the dual basis of $X_*(T) \cong \mathbb{Z}^n$. Then, setting

$$\Delta = \{\varepsilon_i - \varepsilon_{i+1}, 1 \leq i < n\} \text{ and } \Delta^\vee = \{\varepsilon_i^\vee - \varepsilon_{i+1}^\vee, 1 \leq i < n\},$$

we see that $(X^*(T), \Delta, X_*(T), \Delta^\vee) = \psi \cong \psi^\vee$.

On the other hand, one can show that if G is simply connected (resp. adjoint), then $\widehat{G}$ is adjoint (resp. simply connected). Since the map that sends $R(G,T)$ to $R(G,T)^\vee$ permutes the types B_n and C_n, we get that if $G \cong \mathrm{Sp}_{2n}$, then $\widehat{G} \cong \mathrm{SO}_{2n+1}$.

When G is not split it is often useful to work with the following variant of the complex dual: choosing a pinning $(B, T, \{x_\alpha\}_{\alpha \in \Delta(B)})$ of G, we obtain a bijection

$$\mathrm{Aut}(G, B, T, \{x_\alpha\}_{\alpha \in \Delta(B)}) \xrightarrow{\sim} \mathrm{Aut}(\psi).$$

Picking $\gamma \in \Gamma_F$, there exists $g \in G(\overline{F})$ such that ${}^\gamma B = gBg^{-1}$ and ${}^\gamma T = gTg^{-1}$. Therefore we have a map $\Gamma_F \to \mathrm{Aut}(\psi)$. Moreover, since $\mathrm{Aut}(\psi) = \mathrm{Aut}(\psi^\vee)$, choosing a monomorphism $\mathrm{Aut}(\psi^\vee) \to \mathrm{Aut}(\widehat{G}, \widehat{B}, \widehat{T})$ gives a homomorphism $\Gamma_F \to \mathrm{Aut}(\widehat{G}, \widehat{B}, \widehat{T})$. We define the **$L$-group** of G to be ${}^L G = \widehat{G} \rtimes \Gamma_F$.

Now we can give the definition of an L-parameter:

Definition 1.6. An **admissible L-parameter** is a continuous morphism $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ over Γ_F which is:

1. Semisimple on the first factor, i.e. the elements in $\varphi(\mathrm{WD_F} \times 1)$ are semisimple.
2. Algebraic on the second factor i.e. the map

$$\mathrm{SL}_2(\mathbb{C}) \hookrightarrow \mathrm{WD_F} \xrightarrow{\varphi} {}^L G \to \widehat{G}$$

   is a map of algebraic groups.

The set of equivalence classes modulo inner automorphisms by elements of $\widehat{G}$ of admissible L-parameters is denoted by $\Phi(G)$. We say that an L-parameter is **tempered** if its image is bounded in $\widehat{G}$. We will denote the set of tempered L-parameter by $\Phi_{\mathrm{temp}}(G)$.

Example 1.7. If $G = \mathrm{GL}_n$ then an L-parameters is a Weil-Deligne representation of dimension n.

1.2 The conjecture

Now that we have defined all the basic objects, we can state the conjecture. Let $\varphi \in \Phi_{\mathrm{temp}}(G)$. We define the group

$$S_\varphi := \mathrm{Cent}(\varphi(\mathrm{WD_F}), \widehat{G}) = \{g \in \widehat{G} \mid g\varphi(\mathrm{WD_F})g^{-1} \subset \varphi(\mathrm{WD_F})\}.$$

This is an algebraic group with reductive identity component and it contains $Z(\widehat{G})^{\Gamma_F}$. We define

$$\overline{S_\varphi} := S_\varphi / Z(\widehat{G})^{\Gamma_F}.$$

Conjecture (Unrefined LLC): There exists a map $\mathrm{LL} : \Pi_{\mathrm{temp}}(G) \to \Phi_{\mathrm{temp}}(G)$ with finite fibers. We call the fiber of $\varphi \in \Phi_{\mathrm{temp}}(G)$ the ***L*-packet** associated to φ, and we denote it by Π_φ or $\Pi_\varphi(G)$. Moreover, there is a bijection between the set $\Pi_\varphi(G)$ and the set of irreducible representations of the finite group $\pi_0(\overline{S_\varphi})$.

In particular, we are just asking for an explicit description of the L-packets. This conjecture can be further refined in many different ways. One can ask for LL to be in some sense "functorial" or compatible with parabolic induction. Moreover, one might want to generalize this conjecture to non-quasi-split groups. Since these generalizations are not going to be relevant to this thesis, we just refer to [KT] and [Kal16].

The existence of a conjectural map LL like the one that we are looking for is well known for several groups. For example, for $G \cong \mathrm{GL}_1 = \mathbb{G}_m/F$ the map comes from the local class field theory: since $\mathbb{G}_m$ is abelian, the group $\pi_0(\overline{S_\varphi})$ is trivial for every L-parameter φ. Therefore we expect a bijection between smooth irreducible representation and L-parameters. The bijection comes from local class field theory as follows:

$$\mathrm{Hom}(\mathbb{G}_m(F), \mathbb{C}^\times) \cong \mathrm{Hom}(W_F^{\mathrm{ab}}, \mathbb{C}^\times) \cong \mathrm{Hom}(W_F, \mathbb{C}^\times).$$

Similarly, we can extended this argument to get a map for any split torus: if $G \cong T = \mathbb{G}_m^n$ we have

$$\begin{aligned}\mathrm{Hom}(T(F), \mathbb{C}^\times) &\cong \mathrm{Hom}(F^\times \otimes X_*(T), \mathbb{C}^\times) \cong \mathrm{Hom}(F^\times, \mathbb{C}^\times \otimes X_*(\widehat{T}))\\ &\cong \mathrm{Hom}(W_F^{\mathrm{ab}}, \widehat{T}(\mathbb{C})) \cong \mathrm{Hom}(W_F, \widehat{T}(\mathbb{C})).\end{aligned}$$

The existence of a map for any torus is slightly harder to prove. It was already known 60 years ago and it was one of the motivating examples of the theory:

Theorem 1.8 (Langlands). *Let T/F be any torus. Then there exists a bijection* $\mathrm{LL}_T : \Pi(T) \to \Phi(T)$.

Proof. The original proof is due to Langlands [Lan97, Theorem 1], but we will follow [Yu09] for a more recent and elementary exposition. We will just give a sketch of the proof.

First of all, since the elements of $\Phi(T)$ are trivial on SL_2, we have that

$$\Phi(T) = H^1(W_F, \widehat{T}(\mathbb{C})).$$

Let E/F be a finite Galois extension such that T is split over E and consider the restriction map

$$H^1(W_E, \widehat{T}(\mathbb{C})) \to H^1(W_F, \widehat{T}(\mathbb{C})).$$

The existence of a bijection $\mathrm{LL}_T : \Pi(T) \to H^1(W_F, \widehat{T})$ follows from the following three statements:

1. The restriction map $H^1(W_E, \widehat{T}(\mathbb{C})) \to H^1(W_F, \widehat{T}(\mathbb{C}))$ factors through $H^1(W_F, \widehat{T}(\mathbb{C}))_{\Gamma_{E/F}}$.

2. The natural map $\mathrm{Hom}(T(E), \mathbb{C}^\times)_{\Gamma_{E/F}} \to \mathrm{Hom}(T(F), \mathbb{C}^\times)$, which comes from the restriction map $\mathrm{Hom}(T(E), \mathbb{C}^\times) \to \mathrm{Hom}(T(F), \mathbb{C}^\times)$, is an isomorphism.

3. Define the map LL_T to be the composition

$$\mathrm{Hom}(T(F), \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Hom}(T(E), \mathbb{C})_{\Gamma_{E/F}} \xrightarrow{\sim} H^1(W_E, \widehat{T}(\mathbb{C}))_{\Gamma_{E/F}} \xrightarrow{\sim} H^1(W_F, \widehat{T}(\mathbb{C})).$$

   Then LL_T is independent of the choice of E.

The proof of these three claims is in [Yu09, Section 7.7]. □

Notation 1.9. From now on, if T is any torus and $\chi \in \Pi_{\mathrm{temp}}(T)$, we will denote φ_χ the associated L-parameter. Conversely, if φ is an element in $\Phi_{\mathrm{temp}}(T)$, we will denote by χ_φ the associated unramified character.

1.3 The unramified conjecture

Definition 1.10. A **hyperspecial** compact subgroup K of $G(F)$ is a subgroup that is the stabilizer of a hyperspecial vertex in the reduced Bruhat-Tits building $\mathcal{B}(G)$ ([KP23, 7.11.1]). Equivalently, K is hyperspecial if there exists a smooth affine group scheme $\mathcal{G}$ over $\mathcal{O}_F$ such that the following conditions hold:

1. $\mathcal{G}(\mathcal{O}_F) = K$;
2. $\mathcal{G}_F \cong G$;
3. $\mathcal{G}_{\mathfrak{f}}$ is a connected reductive group.

Example 1.11.
- If $G = \mathrm{SL}_2 /\mathbb{Q}_p$ we have 2 conjugacy classes of hyperspecial compact subgroups, with representatives $K = \left(\begin{smallmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{smallmatrix}\right)$ and $K' = \left(\begin{smallmatrix} \mathbb{Z}_p & p\mathbb{Z}_p \\ p^{-1}\mathbb{Z}_p & \mathbb{Z}_p \end{smallmatrix}\right)$. On the other hand, it is a well known fact that GL_n has just one conjugacy class of hyperspecial maximal compact subgroups.
- If G is adjoint then all the hyperspecial maximal compact subgroups are conjugated [KP23, Proposition 10.2.2]. This is going to be important later.

Not every group contains a hyperspecial compact subgroup, since it is not true that every group has a hyperspecial vertex in its Bruhat-Tits building. An example of a group with no hyperspecial maximal compact subgroup is $\mathrm{SL}(D)$ with D a division algebra over F. In this case, $\mathcal{B}(\mathrm{SL}(D))$ is just a single point x. There exists an unramified field extension E/F such that the base change of $\mathrm{SL}(D)$ to E is isomorphic to $\mathrm{SL}_m(E)$ for some $m \in \mathbb{N}$. The base change induces an inclusion of $\mathcal{B}(\mathrm{SL}(D))$ into $\mathcal{B}(\mathrm{SL}_m(E))$ that sends x to the barycenter of a chamber of $\mathcal{B}(\mathrm{SL}_m(E))$. Therefore x is a special point of $\mathcal{B}(\mathrm{SL}(D))$ but it is not hyperspecial.

Definition 1.12. A reductive group G is **unramified** if it is quasi-spit and it splits over an unramified extension. Equivalently, a group is unramified if its Bruhat-Tits building has a hyperspecial vertex [KP23, Remark 7.11.2].

Unless otherwise stated, in this section we will assume G to be unramified.

Definition 1.13. Let K be a hyperspecial maximal compact subgroup of an unramified group $G(F)$. A smooth representation (π, V) of $G(F)$ is called **spherical** with respect to K (or K-spherical) if $V^K \neq 0$. We call (π, V) **unramified** if it is spherical with respect to some hyperspecial maximal compact subgroup. We will denote the set of unramified irreducible representations of $G(F)$ by $\Pi_{\mathrm{ur}}(G)$.

Remark 1.14. The property of being K-spherical for a representation (π, V) depends only on the conjugacy class of K. In fact, it is easy to check that $\pi(g)V^K = V^{gKg^{-1}}$.

Remark 1.15. If K is a hyperspecial maximal compact subgroup, then the bijection coming from Theorem 1.2 restricts to a bijection between irreducible K-spherical representations of $G(F)$ and simple $\mathcal{H}(G(F), K)$-modules.

Example 1.16. The simplest examples of unramified representations are characters of a split torus $T(F)$ trivial on $T(\mathcal{O}_F)$. More general, if T is any torus, a character χ of $T(F)$ it is unramified if it is trivial on the maximal compact subgroup $T(F)^1$. We can see $T(F)^1$ as the kernel of the valuation map

$$\omega_T : T(F) \to X_*(T)$$

defined by the property $\langle \omega_T(t), \alpha \rangle = |\alpha(t)|_F$.

If $T = \mathbb{G}_m$, then the map ω_T is just the valuation map. This is a simple check: let $x \in F^\times$ and let $f = n \in X^*(T) \cong \mathbb{Z}$. Then

$$n \cdot \omega_T(x) = \langle \omega_T(x), f \rangle = \mathrm{val}(f(x)) = \mathrm{val}(x^n) = n \cdot \mathrm{val}(f(x)),$$

so $\omega_T(x) = \mathrm{val}(x)$. Then $\mathbb{G}_m(F)^1 = \mathcal{O}_F^\times$. Generalizing this argument, we get that $T(F)^1 = T(\mathcal{O}_F)$ for a general split torus T.

For $T = \mathbb{G}_m$ the unramified characters correspond to characters of the Galois group trivial on the inertia via the local reciprocity map. This motivates the term unramified and the following definition:

Definition 1.17. An L-parameter φ of G is **unramified** if it is trivial on $\mathrm{SL}_2(\mathbb{C})$ and on I_K. We will denote the set of unramified L-parameters of G by $\Phi_{\mathrm{ur}}(G)$.

The LLC for tori restricts to a correspondence between unramified L-parameters and unramified representation:

Theorem 1.18. *Let T be an unramified torus. Then LL_T induces a bijection $\mathrm{LL}_{\mathrm{ur}} : \Pi_{\mathrm{ur}}(T) \to \Phi_{\mathrm{ur}}(T)$.*

Proof. For the proof we follow [Bor79, Section 9.5]. Let T/F be an unramified torus that splits over an unramified extension F'/F.

From Hilbert's Theorem 90, we know that $H^1(\Gamma_{F'/F}, \mathcal{O}_{F'}^\times) = 1$, and, since T splits over F' it follows that $H^1(\Gamma_{F'/F}, T(F')^1) = 1$. Using the short exact sequence

$$0 \to T(F')^1 \to T(F') \to T(F')/T(F')^1 \to 0,$$

by the long exact sequence of Galois cohomology, we get $(T(F')/T(F')^1)^{\Gamma_F} \cong T(F)/T(F)^1$, and therefore

$$(T(F')/T(F')^1)^{\Gamma_F} \cong (X^*(T))^{\Gamma_F}.$$

This argument gives a more explicit description of the unramified L-packets:

$$\Pi_{\mathrm{ur}}(T) \cong \mathrm{Hom}(X_*(T)^{\Gamma_F}, \mathbb{C}^\times).$$

The RHS consists of the $\mathbb{C}$-point of the Langlands dual of a maximal split torus T_d in T. From [Bor79, Lemma 6.4] we get that $\Pi_{\mathrm{ur}}(T)$ is in bijection with $(\widehat{T} \rtimes \mathrm{frob})/\operatorname{Inn}(\widehat{T})$, where $\operatorname{Inn}(\widehat{T})$ is the group of inner automorphisms of $\widehat{T}$.

On the other hand, every unramified L-parameter φ of T, is determined by by the image of the Frobenius, up to conjugation by $\widehat{T}$. So we have

$$\Phi_{\mathrm{ur}}(T) \cong (\widehat{T} \rtimes \mathrm{frob})/\operatorname{Inn}(\widehat{T})$$

and therefore we have the bijection

$$\mathrm{LL}_{\mathrm{ur}} : \Pi_{\mathrm{ur}}(T) \to \Phi_{\mathrm{ur}}(T).$$

□

The importance of unramified representations lies in their relevance in the global Langlands conjecture since one can show that automorphic representations are unramified at almost every place. A fundamental tool to study unramified representation is the Satake isomorphism, which gives us a link between unramified representations of an unramified group and unramified representations of one of its maximal tori. For the proof, we will follow [Car79, Theorem 4.1] filling in some omitted details using some facts from [HR10].

We will need the following important results:

Theorem 1.19 (Cartan decomposition)**.** *Let S be a maximal split torus of G, x a special vertex in $\mathcal{A}(S)$ and $\mathcal{P}_x$ the associated parahoric subgroup with type W_x. Then*

$$G(F) = \bigsqcup_{w \in W_x \backslash \widetilde{W} / W_x} \mathcal{P}_x w \mathcal{P}_x.$$

Proof. For a general introduction to Tits systems, we refer to [Bou02, Chapter 4,section 2]. The statement of this theorem is much more general and it holds for a generic Tits system: given a Tits system (G, B, N, R) with Weyl group W, for every two standard parabolic subgroups P, P' of type W_P and $W_{P'}$ respectively, and for every $w \in W$, we have

$$PwP = BW_P w W_{P'} B.$$

In particular, we have a bijection

$$P\backslash G/P' \longleftrightarrow W_P\backslash W/W_{P'}.$$

The proof of this fact is in [Bou02, Chapter 4, Section 2, Subsection 5, Proposition 2]. Applying this to the Iwahori-Tits system proves the proposition for $G(F)^0$. From this, one can generalize to the case for $G(F) \neq G(F)^0$ [KP23, Theorem 5.2.1]. □

Theorem 1.20 (Satake isomorphism)**.** *Let S a maximal split torus of an unramified group G, let T be the centralizer of S in G, and let $B \supset T$ be a Borel subgroup with unipotent radical U. Let K be a maximal hyperspecial compact subgroup of $G(F)$, that is the stabilizer of a hyperspecial vertex in $\mathcal{A}(S)$. We normalize the Haar measure μ on $G(F)$ such that $\mu(K) = 1$. Then the map*

$$\mathscr{S} : \mathcal{H}(G(F), K) \to \mathcal{H}(T(F), T(F) \cap K)^W$$

that sends

$$f \mapsto \big(t \mapsto \mathscr{S} f(t) := \delta(t)^{-1/2} \int_{U(F)} f(ut)du,$$

is an isomorphism of $\mathbb{C}$-algebras. This map is called the ***Satake transform*** *or the* ***Satake isomorphism****.*

Proof. We divide the proof into steps:

1. $\mathscr{S}$ is an algebra homomorphism from $\mathcal{H}(G(F), K)$ to $\mathcal{H}(T(F), T(F) \cap K)$;

2. The image of $\mathscr{S}$ is contained $\mathcal{H}(T(F), T(F) \cap K)^W$;

3. $\mathscr{S}$ is a linear isomorphism.

For the first part, we notice that $\mathscr{S}$ can be written as the composition

$$\mathcal{H}(G(F), K) \xrightarrow{\alpha} \mathcal{H}(B(F)) \xrightarrow{\beta} \mathcal{H}(T(F)) \xrightarrow{\gamma} \mathcal{H}(T(F)),$$

where α is just the restriction from $G(F)$ to $B(F)$, β sends f to $\beta f(t) = \int_{U(F)} f(tu)du$, and γ is multiplication by $\delta^{-1/2}$. Let $d_l x$ be a left invariant Haar measure on $B(F)$. The map α is an algebra homomorphism, since if $b \in B(F)$, then

$$\begin{aligned}
\big(f_1 * f_2\big)(b) &= \int_{G(F)} f_1(g) f_2(g^{-1}b)\, dg = \int_K \int_{B(F)} f_1(xk) f_2(k^{-1}x^{-1}b)\, d_l x\, dk \\
&= \int_K \int_{B(F)} f_1(x) f_2(x^{-1}b)\, d_l x\, dk = \int_{B(F)} f_1(x) f_2(x^{-1}b)\, d_l x \\
&= \big(f_1|_{B(F)} * f_2|_{B(F)}\big)(b),
\end{aligned}$$

for every $f_1, f_2 \in \mathcal{H}(G(F), K)$. In the second equality, we use the Iwasawa decomposition $G(F) = BK$, in the third we use that f_1 and f_2 are K-bi-invariant, and in the fourth we use that we chose a Haar measure such that the measure of K is 1.

Now we want to show that β is an algebra homomorphism. We know that

$$\beta(f_1 * f_2)(t) = \int_{U(F)} \int_{B(F)} f_1(x) f_2(x^{-1}ut)\, d_l x\, du$$

for every $f_1, f_2 \in \mathcal{H}(B(F))$, and we need to prove that this expression is equal to

$$\int_{T(F)} \left(\int_{U(F)} f_1(ux)\, du \cdot \int_{U(F)} f_2(u'x^{-1}t)\, du' \right) dx,$$

with dx an Haar measure on $T(F)$. The equality follows from

$$\int_{T(F)} \left(\int_{U(F)} f_1(ux)\, du \cdot \int_{U(F)} f_2(u'x^{-1}t)\, du' \right) dx = \int_{T(F)} \int_{U(F)} \int_{U(F)} f_1(ux) f_2(u'x^{-1}t)\, du\, du' dx =$$
$$= \int_{B(F)} \int_{U(F)} f_1(b) f_2(\bar{u}b^{-1}t)\, db\, d\bar{u} = \beta(f_1 * f_2)(b).$$

In the last step, we are using the change of variables $b = ux$.

Finally, the fact that γ is an algebra homomorphism comes from the fact that δ is a character. In fact,

$$(\delta^{-1/2} f_1 * \delta^{-1/2} f_2)(t) = \int_{T(F)} \delta^{-1/2}(x) f_1(x) \delta^{-1/2}(x^{-1}t) f_2(x^{-1}t) dx =$$
$$= \delta(t)^{-1/2} \int_{T(F)} f_1(x) f_2(x^{-1}t) dx = \delta(t)^{-1/2} \big(f_1 * f_2\big)(t).$$

This proves that $\mathscr{S}$ is an algebra homomorphism.

Now we start with the proof of the second part. First we point out that

$$W \cong N(S)/S \cong N(S) \cap K / S \cap K.$$

In fact, if $n \in N(S)$ acts on $\varphi_x \in \mathcal{A}(S)$ via $n \cdot \varphi_{\alpha,x}(-) = \varphi_{n^{-1}\alpha,x}(n^{-1} - n)$, we can choose $s \in S$ such that $(ns) \cdot \varphi_x = \varphi_x$. Therefore ns stabilizes x. So $nS \in W$ has a lift in $N(S) \cap K$.

Since $W \cong N(S) \cap K/S \cap K$, the image of $\mathscr{S}$ is contained in the Weyl invariants if and only if

$$\mathscr{S} f(xtx^{-1}) = \mathscr{S} f(x), \tag{1}$$

for $t \in T(F)$ and $x \in N(S) \cap K$. Now we consider the function $t \mapsto \det(\mathrm{Ad}_{\mathfrak{u}}(t) - 1)$ from $T(F)$ to F. Since this is a polynomial and it is nonzero, the elements of $T(F)$ that do not annihilate this function are dense in $T(F)$. We call these elements **regular**. Hence by continuity it suffices to prove 1 just for t regular.

Using [Car79, Lemma 4.1] we get that

$$\mathscr{S} f(t) = D(t) \int_{G \backslash T} f(gtg^{-1})\, dg,$$

where $D(t) = \delta(t)^{-1/2} \cdot |\det(\mathrm{Ad}_{\mathfrak{u}}(t) - 1|$. We can compute

$$\begin{aligned} D(t)^2 &= |\det(\mathrm{Ad}_{\mathfrak{u}}(t) - 1)|_F^2 \cdot |\det(\mathrm{Ad}_{\mathfrak{u}}(t))|_F^{-1} \\ &= |\det(\mathrm{Ad}_{\mathfrak{u}}(t) - 1)|_F \cdot \left|\det(\mathrm{Ad}_{\mathfrak{u}}(t^{-1}) - 1)\right|_F \\ &= |\det(\mathrm{Ad}_{\mathfrak{u}}(t) - 1)|_F \cdot |\det(\mathrm{Ad}_{\mathfrak{u}^-}(t) - 1)|_F . \end{aligned}$$

Here $\mathfrak{u}^-$ is the nilpotent subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{u}$ relative to $\mathfrak{t}$, so that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$. The last equality holds because the weights in $\mathfrak{u} \otimes_F \overline{F}$ are the inverses of the weights of $\mathfrak{u}^- \otimes_F \overline{F}$.

From the decomposition of $\mathfrak{g}$ that we stated, we get that

$$D(t) = \left|\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{t}}(t) - 1)\right|_F^{1/2} .$$

Therefore D is invariant under precomposition with conjugation by elements of $N(T)$.

Let now $x \in N(T) \cap K$, $f \in \mathcal{H}(G(F), K)$ and $t \in T(F)$ regular. The element x acts by inner automorphism on G and T and leaves invariant their Haar measures. Therefore, it leaves invariant the Haar measure on G/T. Hence

$$\int_{G \backslash T} f(g(xtx^{-1})g^{-1})\, dg = \int_{G \backslash T} f((x^{-1}gx)t(x^{-1}g^{-1}x))\, dg = \int_{G \backslash T} f(gtg^{-1})\, dg,$$

where there last inequality comes from the fact that f is K bi-invariant. So we proved that the image of $\mathscr{S}$ is in $\mathcal{H}(T(F), T(F)\cap K)^W$.

What is left to show is that $\mathscr{S}$ is an actual bijection. We know that a basis of $\mathcal{H}(G(F), K)$ is given by the characteristic functions on the double cosets of K. From the Cartan decomposition, we get that a basis is given by

$$\{\chi_{KvK}\}_{v\in W_K\backslash\widetilde{W}/W_K},$$

where W_K is the type of K as a parahoric subgroup of $G(F)$. We are going to denote χ_{KvK} just by χ_v.

The embedding of T in $N_G(T)$, induces a bijection between $W_K\backslash\widetilde{W}/W_K$ and the set $W\backslash T(F)/T(\mathcal{O}_F)$ [HR10, Theorem 1.0.3]. From now on we are going to identify this two sets under this bijection. We denote the characteristic function $\chi_{T(F)\cap KvT(F)\cap K}$ on a K double coset of $T(F)$ by χ'_v. A basis of $\mathcal{H}(T(F), T(F)\cap K)^W$ is given by

$$\left\{b_v := \frac{1}{|W(v)|}\sum_{w\in W}\chi'_{wvw^{-1}}\right\}_{v\in W\backslash T(F)/T(\mathcal{O}_F)},$$

where $W(v)$ is the stabilizer of v in W.

We define the infinite matrix $\{c(v,v')\}_{v,v'\in W_K\backslash\widetilde{W}/W_K}$ by

$$\mathscr{S}\chi_{v'} = \sum_{v\in W_K\backslash\widetilde{W}/W_K} c(v,v')b_v.$$

Then, if $t\in KvK$ we have

$$c(v,v') = \mathscr{S}\chi_{v'}(t) = \delta(t)^{-1/2}\mu(Kv'K\cap UvK).$$

$Kv'K\cap UvK$ is empty unless $v'-v$ is a linear combination with non-negative real coefficients of positive roots. We can choose a suitable lexicographic order so that $c(v,v') = 0$ unless $0\geq v'\geq v$. Moreover, $KvK\cap UvK\supset tK$, so $c(v,v)\geq\delta(t)^{-1/2}$ and it is not 0. Therefore our infinite matrix is "upper triangular" and $\mathscr{S}$ is a bijection. For further details about the last paragraph we refer to [Car79, Theorem 4.1], for a more general and recent approach we refer to [HR10]. □

Thanks to the Satake isomorphism, we can now construct a map $\mathrm{LL}_{\mathrm{ur}}:\Pi_{\mathrm{ur}}(G)\to\Phi_{\mathrm{ur}}(G)$. Let $(\pi,V)\in\Pi_{\mathrm{ur}}(G)$ be a K-spherical representation with K a hyperspecial maximal compact subgroup of $G(F)$. Then V is an $\mathcal{H}(G,K)\cong\mathcal{H}(T(F), K\cap T(F))^W$-module for some maximal torus T in G, so $\mathscr{S}(\pi,V)$ is an unramified representation of T. We can now apply LL_T 1.18 to get an unramified L-parameter φ_π of T, and since ${}^LT\subset{}^LG$, we can view φ_π as an unramified L-parameter of G. We define $\mathrm{LL}_{\mathrm{ur}}$ to be the map $\pi\mapsto\varphi_\pi$.

The unramified L-packets Now we want to describe the L-packet $\Pi_{\varphi_\chi}(G) = \Pi_{\varphi_\chi} = \mathrm{LL}_{\mathrm{ur}}^{-1}(\varphi_\chi)$ for χ an unramified character of a maximal torus T of G (here we are seeing φ_χ as an L-parameter of G via ${}^LT\subset{}^LG$). We will follow [Key87] for an introduction on the R-group. Some non-trivial harmonic analysis on p-adic reductive groups is used, and, since this is not going to be relevant to this thesis, we are just going to reference [Sil79] or [Sil78].

We start by saying what is known for GL_n:

Proposition 1.21. *The Satake isomorphism induces a bijection between* $\Pi_{\mathrm{ur}}(\mathrm{GL}_n)$ *and* $\Phi_{\mathrm{ur}}(\mathrm{GL}_n)$.

Proof. First we recall that in $\mathrm{GL}_n(F)$ there is only one hyperspecial maximal compact subgroup up to conjugacy, and a representative is given by $K = \mathrm{GL}_n(\mathcal{O}_F)$. Let T be the torus consisting of diagonal matrices. Then the Hecke algebra $\mathcal{H}(T(F), T(\mathcal{O}_F))$ is isomorphic to $\mathbb{C}[X_*(T)]$ (to show this is enough to notice that the valuation map ω_T is surjective in the case of split tori).

The dual torus $\widehat{T}$ is $\mathrm{Spec}(\mathbb{C}[X_*(T)])$. Then homomorphisms $\mathbb{C}[X_*(T)]^W \to \mathbb{C}$ correspond to W-conjugacy classes of elements in $\widehat{T}(\mathbb{C})$. These conjugacy classes, in turn, are in bijection with the conjugacy classes of diagonalizable elements in $\mathrm{GL}_n(\mathbb{C})$. Finally, these conjugacy classes are in bijection with the unramified L-parameters, since these are determined just by the image of Frobenius up to conjugacy. At the same time, by the Satake isomorphism, maps $\mathbb{C}[X_*(T)]^W \to \mathbb{C}$ correspond to maps $\mathcal{H}(\mathrm{GL}_n(F), \mathrm{GL}_n(\mathcal{O}_F)) \to \mathbb{C}$, and therefore to irreducible $\mathcal{H}(\mathrm{GL}_n(F), \mathrm{GL}_n(\mathcal{O}_F))$-modules. This gives us a bijection $\Pi_{\mathrm{ur}}(\mathrm{GL}_n) \to \Phi_{\mathrm{ur}}(\mathrm{GL}_n)$. $\square$

The case of GL_n is the easiest one after the case of tori. In fact, it is not generally true that $\Pi_{\mathrm{ur}}(G)$ and $\Phi_{\mathrm{ur}}(G)$ are in bijection for a general group G. The reason is that two different representations that are unramified for two different classes of hyperspecial maximal compact subgroups might correspond to the same L-parameter.

For the rest of this section, we are going to denote $S \subset G$ a split maximal torus, $T := \mathrm{Cent}_G(S) \subset G$ and we fix a Borel subgroup $B = TU$ containing T with unipotent radical U.

Definition 1.22. Let χ be an unramified character of $T(F)$. Then we call the normalized parabolic induction $I(\chi) := i_B^G\chi$ the **unramified principal series** associated to χ.

The unramified principal series of an unramified character χ is the fundamental tool to understand the L-packet Π_{φ_χ}. Note that if we fix K a hyperspecial maximal compact subgroup of $G(F)$ that is the stabilizer of a vertex in $\mathcal{A}(S)$, then we can use the Iwasawa decomposition $G(F) = B(F)K$ to see that the fixed-point set of K in $I(\chi)$ is one dimensional. It follows that $I(\chi)$ has at most one irreducible subquotient with nonzero fixed vectors under K.

Remark 1.23. We have an equality between Π_{φ_χ} and the set of irreducible unramified subquotients of $I(\chi)$. This comes as a corollary of the description of unramified representation via spherical functions [Car79, Section 4]. Let K be a hyperspecial maximal compact subgroup of $G(F)$ that is the stabilizer of a hyperspecial vertex in $\mathcal{A}(S)$. Then every K-spherical representation V is of the form V_Γ for some K-spherical function Γ [Car79, Theorem 4.3], and every K-spherical function Γ is of the form $\Gamma_{\chi'}$ for some K-unramified character χ' of T [Car79, Theorem 4.2]. Finally it is shown that every representation V_{Γ_χ} appears as a subquotient of $I(\chi)$ and it is the unique K-spherical subquotient.

Proposition 1.24. *Let χ be any unramified character of $T(F)$.*

1. *If χ is unitary and regular (i.e. $w \cdot \chi \neq \chi$ for every $w \in W$), then $I(\chi)$ is irreducible.*
2. *Let $w \in W$. The representations $I(\chi)$ and $I(w \cdot \chi)$ have the same Harish-Chandra character, hence they are isomorphic if they are irreducible.*
3. *The $\mathcal{H}(G)$-module $I(\chi)$ is of finite length.*

Proof. The proof can be found in [Car79, Theorem 3.3]. $\square$

Now we are going to introduce the tool that is going to connect Π_{φ_χ} to $\mathrm{Irr}(\pi_0(\overline{S_{\varphi_\chi}}))$: the R-group.

Let $\overline{B} = T\overline{U}$ be the Borel subgroup opposite to B with respect to T, with unipotent radical $\overline{U}$. Then we can define the intertwining operators

$$\boldsymbol{A}(w, \chi) : I(\chi) \to I(w \cdot \chi)$$
$$\boldsymbol{A}(w, \chi)(f)(g) := \int_{U(F)\cap w\overline{U}(F)w^{-1}} f(guw)du \quad \forall g \in G,$$

for ever $w \in W$ and for χ belonging to an appropriate set of cocharacters. Moreover, they can be analytically continued so that they are defined for every unitary character. These operators satisfy the cocycle condition

$$\boldsymbol{A}(w_1w_2, \chi) = \boldsymbol{A}(w_1, w_2\chi)A(w_2, \chi)$$

if w_1, w_2 satisfy
$$l(w_1w_2) = l(w_1) + l(w_2);$$
here $l(w)$ is the length of w in the Coxeter group W.

We now define the normalized intertwining operators
$$\mathbb{A}(w,\chi) := \frac{1}{c_w(\chi)}\boldsymbol{A}(w,\chi)$$
where c_w is the Harish-Chandra c-function defined as in [Sil79] or [Wal75, Section 7]. We define
$$W_\chi := \{w \in W \mid w \cdot \chi = \chi\}.$$

Theorem 1.25 (Harish-Chandra)**.** *Let χ be a unitary character of $T(F)$. Then the algebra* $\mathrm{End}(I(\chi))$ *is spanned by the operators*
$$\{\mathbb{A}(w,\chi) \mid w \in W_\chi\}.$$

Proof. The proof can be found in [Sil79, Theorem 5.5.3.2]. □

Remark 1.26. Notice that if χ is regular, then W_χ is trivial and $\mathrm{End}(I(\chi))$ is generated by $\mathbb{A}(1,\chi)$, showing that $I(\chi)$ is irreducible.

We define now $W' = W'_\chi$ to be the subgroup of W_χ for which $\mathbb{A}(w,\chi)$ is a scalar. Moreover we define $\Delta'_\chi := \{\alpha \text{ root} \mid s_\alpha \in W'_\chi\}$.

Theorem 1.27 (Knapp-Stein dimension theorem)**.** *Suppose χ is a unitary character of $T(F)$. Then the dimension of the commuting algebra* $\mathrm{End}(I(\chi))$ *is* $[W_\chi : W'_\chi]$.

Proof. The proof is [Sil78]. □

We can define the R-group:
$$R = R_\chi := \{w \in W_\chi \mid \alpha > 0 \text{ and } \alpha \in \Delta'_\chi \text{ imply that } wa > 0\}.$$
Then
$$W_\chi = W' \rtimes R_\chi.$$

This means, from Harish-Chandra's theorem 1.25 and the dimension theorem 1.27 that the intertwining operators
$$\{\mathbb{A}(w,\chi)\}_{w\in R_\chi}$$
form a linear basis of $\mathrm{End}(I(\chi))$.

By Schur's Lemma, the intertwining operators satisfy the cocycle relation with no condition on the lengths of the Weyl group elements, up to a scalar. We define a 2-cocycle η of the Weyl group by
$$\mathbb{A}(w_1w_2,\chi) = \eta(w_1,w_2)\mathbb{A}(w_1, w_2\cdot\chi)\mathbb{A}(w_2,\chi).$$
Then the commuting algebra $\mathrm{End}(I(\chi))$ is isomorphic to the group algebra $\mathbb{C}_\eta[R_\chi]$, with multiplication twisted by the 2-cocycle η.

Theorem 1.28. *Suppose that the intertwining operators corresponding to the simple reflections $\{s_\alpha\}_{\alpha\in R(G,T)}$ are normalized so that*
$$\mathbb{A}(s_\alpha, s_\alpha\chi)\mathbb{A}(s_\alpha,\chi) = \mathrm{Id}\,.$$
Then $\eta \equiv 1$, i.e. the cocycle relation holds with no condition on the lengths of the Weyl group elements. Moreover, this normalization is always possible.

Proof. This was shown by Keys in [Key82a, Chapter 1, Section 2, Theorem 2]. ☐

Proposition 1.29. *Suppose χ is a unitary character of $T(F)$. Then:*

1. *The commuting algebra* $\mathrm{End}(I(\chi))$ *is isomorphic to the group algebra* $\mathbb{C}[R]$*;*
2. *$I(\chi)$ decomposes with multiplicity one if and only if R_χ is abelian;*
3. *The inequivalent irreducible components π_i of the representation $I(\chi)$ of G are parametrized by the irreducible representations $\rho_i = \rho(\pi_i)$ of R_χ;*
4. *The multiplicity with which a component π_i occurs in $I(\chi)$ is equal to the dimension of the representation ρ_i which parametrizes it.*

Remark 1.30. In [Key82a] Keys showed that R_χ is always abelian for types B_n, C_n, E_6, F_4 and G_2, but for other types, it can be non-abelian. But if we assume that χ is unramified (and this is the case we are interested in) then R_χ is always abelian. This was shown by Keys in [Key82b] if G is simply connected, almost simple and semi-simple, and by Mishra [Mis13, Corollary 10] for general G.

Proof. The proof of this can be found in [Key87]. The first two statements are clear since $\eta \cong 1$, and third and fourth come from results on the group algebra of a finite group. ☐

Therefore we have a connection between Π_{φ_χ} and the irreducible representation of R_χ. What we need now is a bijection between R_χ and $\pi_0(\overline{S_{\varphi_\chi}})$. We will first need the following lemma:

Lemma 1.31. *Let χ a unitary character of $T(F)$. Let A be the connected component of the identity in the Weil group invariants $\widehat{T}^{W_F}$ of $\widehat{T}$ and let N be the normalizer of $\widehat{T}$ in $\widehat{G}$. Then:*

1. *The centralizer of A in $\widehat{G}$ is $\widehat{T}$;*
2. *A is a maximal torus of $S^\circ_{\varphi_\chi}$;*
3. *$\widehat{T} \cap S_{\varphi_\chi} = A \cdot Z(\widehat{G})^{W_F}$;*
4. *$\widehat{T} \cap S^\circ_{\varphi_\chi} = A$;*
5. *$N \cap S_{\varphi_\chi}$ is the normalizer of A in S_{φ_χ}.*

Proof. This is [Key87, Lemma 2.5] ☐

Proposition 1.32. *Let χ a unitary character of $T(F)$. Let A and N as in Lemma 1.31. Then the following sequence is exact:*

$$1 \to (N \cap S^\circ_{\varphi_\chi})/A \to (N \cap S_{\varphi_\chi})/A \cdot Z(\widehat{G})^{W_F} \to \pi_0(\overline{S_{\varphi_\chi}}) \to 1.$$

Moreover, the middle term in the exact sequence can be identified with the stabilizer W_χ of χ in the Weyl group of G, and $\pi_0(\overline{S_{\varphi_\chi}}) \cong R_\chi$.

Proof. We follow [Key87, Theorem 2.6]. The fact that the sequence makes sense comes from Lemma 1.31. To prove that the second map is surjective, let $s \in S_{\varphi_\chi}$. This element normalizes $S^\circ_{\varphi_\chi}$ and sAs^{-1} is a maximal torus of $S^\circ_{\varphi_\chi}$. So, since all maximal tori in $S^\circ_{\varphi_\chi}$ are conjugate, there is an element $t \in S^\circ_{\varphi_\chi}$ such that $sAs^{-1} = tAt^{-1}$. Therefore $t^{-1}s$ normalizes A and thus $\widehat{T}$. This means that $t^{-1}s$ has the same image as s. The injectivity of the first map and the exactness in the middle are clear.

What we need to show is that the middle term can actually be identified with W_χ.

From the Langlands correspondence for tori, one sees that $W_\chi = \{w \in W \mid w \cdot \varphi_\chi \text{ is equivalent to } \varphi\}$. Let $w \in W$ and $n \in N(\mathbb{C})$ a representative for w, where w is seen as W_F-invariant element of the Weyl group of $\widehat{T}$. From [Bor79, Lemma 6.2] we know that $w \cdot \varphi_\chi$ is equivalent to $\mathrm{Inn}(n) \circ \varphi_\chi$; therefore w fixes χ if and only if $\mathrm{Inn}(n) \circ \varphi_\chi$ is equivalent to φ_χ. This means that there is $t \in \widehat{T}(\mathbb{C})$ such that $tn \in S_{\varphi_\chi}$. Therefore

$$W_\chi = (N \cap S_{\varphi_\chi})\widehat{T}/\widehat{T} = (N \cap S_{\varphi_\chi})/A \cdot Z(\widehat{G})^{W_F}.$$

Since the first term of the sequence is the Weyl group $W(S^\circ_{\varphi_\chi}, A)$, then $\pi_0(\overline{S_{\varphi_\chi}}) = W_\chi/W(S^\circ_{\varphi_\chi}, A)$.

What is left to do is to identify $W(S^\circ_{\varphi_\chi}, A)$ with the subgroup W'_χ of W_χ, but we omit the proof of this fact. To prove this Keys uses an explicit description of the roots in W'_χ using Plancherel factors. □

Remark 1.33. The last theorem gives us the bijection between Π_{φ_χ} and $\mathrm{Irr}(\pi_0(\overline{S_{\varphi_\chi}}))$. Namely, Π_{φ_χ} is in bijection with the characters of R_χ by Proposition 1.29, and $\mathrm{Irr}(\pi_0(\overline{S_{\varphi_\chi}}))$ is in bijection with the same set by Proposition 1.32. However, this bijection does not tell us which character of R_χ corresponds to which representation in Π_{φ_χ}. An answer to this question is given in [Mis13, Theorem 1], where Mishra, given a character ρ of R_χ, specifies the various hyperspecial subgroups K for which the K-spherical subquotients of $I(\chi)$ corresponds to ρ.

2 The local Langlands conjecture for disconnected groups

In this chapter we state the local Langlands conjecture for disconnected reductive groups, following [Kal22]. By a disconnected reductive group, we mean a disconnected algebraic group whose identity component is reductive. We restrict our attention to the groups $\widetilde{G}$ such that there exists an isomorphism of $\overline{F}$-groups

$$\widetilde{G} \cong G \rtimes A,$$

where G/F is a connected reductive group and A is a finite group acting on G by automorphisms which preserve a fixed $\overline{F}$-pinning. If G is a connected (quasi)split reductive group over F and A acts on G preserving an F-pinning, then we say that the F-group $G \rtimes A$ is a (quasi)split disconnected reductive group.

Example 2.1. The easiest example of a disconnected reductive group is the normalizer of the standard torus T of diagonal matrices in GL_n. Indeed, we have a splitting short exact sequence

$$1 \to T \to N_{\mathrm{GL}_n}(T) \to N_{\mathrm{GL}_n}(T)/T \cong S_n \to 1,$$

with the splitting $S_n \to \mathrm{GL}_n$ given by the permutation matrices in GL_n. Therefore, $N_{\mathrm{GL}_n}(T) \cong T \rtimes S_n$.

An example of a disconnected group that is not of the form $G \rtimes A$ is the normalizer of the torus T' of diagonal matrices inside SL_2. The exact sequence

$$1 \to T' \to N_{\mathrm{SL}_2}(T') \to S_2 \to 1$$

does not split.

From now untill the end of this thesis, we fix a quasi-split disconnected reductive group $\widetilde{G} \cong G \rtimes A$ with an A-fixed pinning $(B, T, \{x_\alpha\}_\alpha)$ of G. We denote $\widetilde{B} := B \rtimes A$ and $\widetilde{T} := T \rtimes A$. Let $\widehat{G}$ be the complex dual of G. We fix the Γ_F-invariant pinning $(\widehat{B}, \widehat{T}, \{\widehat{x_\alpha}\}_\alpha)$ of $\widehat{G}$ dual to the A-fixed pinning of G. We let the group A act on $\widehat{G}$ by fixing this pinning. This means that the action of A on $\widehat{G}$ comes from the isomorphism $\mathrm{Aut}(G, B, T) \cong \mathrm{Aut}(\widehat{G}, \widehat{B}, \widehat{T})$. More precisely, given an element $a \in A$, we have the automorphism a_* of $X_*(T)$ given by $(a_*\lambda)(x) = a(\lambda(x))$ for every $x \in \mathbb{G}_m$ and every $\lambda \in X_*(T)$. We let $a \in A$ act on $\widehat{T}(\mathbb{C}) = \mathrm{Hom}(X_*(T), \mathbb{C}^\times)$ by $(a \cdot t)(\lambda) = t(a_*^{-1}\lambda)$ for $t \in \widehat{T}(\mathbb{C})$ and $\lambda \in X_*(T)$.

We can now begin to formulate a local Langlands conjecture for quasi-split disconnected groups. First we need to define which class of representations of $\widetilde{G}$ we are interested in.

Definition 2.2. A representation of $\widetilde{G}$ is called G**-tempered** if its restriction to G contains a tempered representation. We denote the set of G-tempered irreducible smooth representations of $\widetilde{G}$ by $\Pi_{\text{temp}}(\widetilde{G})$.

In [Kal22, Section 4.6], Kaletha proposes to parametrize G-tempered representation of $\widetilde{G}$ with pairs (φ, ρ) where φ is again an L-parameter $\varphi : \mathrm{WD_F} \to {}^LG$, and ρ is again an irreducible representation of a finite group. The differences with the connected case are the equivalence relation on the set of L-parameters, and consequently, the finite group that should parametrize the L-packets.

First of all, two L-parameters are going to be $\widetilde{G}$-equivalent if they are $\widehat{G} \rtimes A$-conjugate. We are going to denote the set of L-parameters up to $\widetilde{G}$-equivalence by $\Phi(\widetilde{G})$. Secondly, given an L-parameter φ, we define

$$\widetilde{S_\varphi} := \mathrm{Cent}(\varphi, \widehat{G} \rtimes A) = \mathrm{Cent}(\varphi(\mathrm{WD_F}), \widehat{G} \rtimes A).$$

Notice that from the split short exact sequence

$$1 \to G \to \widetilde{G} \to A \to 1$$

we get the short exact sequence

$$1 \to S_\varphi \to \widetilde{S_\varphi} \to A^\varphi \to 1, \tag{2}$$

where A^φ is the stabilizer of the $\widehat{G}$-conjugacy class of φ in A. Moreover, this sequence leads to the exact sequence

$$1 \to \pi_0(S_\varphi) \to \pi_0(\widetilde{S_\varphi}) \to A^\varphi \to 1. \tag{3}$$

Conjecture (Disconnected unrefined LLC): There exists a map $\widetilde{\mathrm{LL}} : \Pi_{\text{temp}}(\widetilde{G}) \to \Phi_{\text{temp}}(\widetilde{G})$ where $\widetilde{\Phi}_{\text{temp}}(\widetilde{G})$ is the set of equivalence classes of tempered L-parameter. We require $\widetilde{\mathrm{LL}}$ to have finite fibers. Moreover, if we denote the fiber of $\varphi \in \Phi_{\text{temp}}(\widetilde{G})$ by $\widetilde{\Pi}_\varphi$, we require $\widetilde{\Pi}_\varphi$ to be in bijection with the set $\mathrm{Irr}(\pi_0(\widetilde{S_\varphi}), \mathrm{Id})$ of those irreducible representations of $\pi_0(\widetilde{S_\varphi})$ which restricted to $\pi_0(Z(\widehat{G})^{\Gamma_F})$ contain the identity.

As in the connected case, this conjecture has been further refined to the case of inner forms of quasi-split disconnected groups. If the reader is interested, the refined conjecture can be found in [Kal22, Conjecture 4.2, Conjecture 5.12]. Moreover, Kaletha conjectures a compatibility between the connected and the disconnected correspondence. In particular, he asks for a G-tempered representation of $\widetilde{G}(F)$ to be in $\widetilde{\Pi}_\varphi$ if and only if its restriction to $G(F)$ intersects Π_φ. This can be found in [Kal22, Conjecture 7.1, Remark 7.2].

Remark 2.3. We want to point out that the classical local Langlands conjecture could have still been phrased using the notation $\mathrm{Irr}(\pi_0(S_\varphi), \mathrm{Id})$ instead of $\mathrm{Irr}(\overline{S_\varphi})$. Here by $\mathrm{Irr}(\pi_0(S_\varphi), \mathrm{Id})$ we mean the set of irreducible representations of $\pi_0(S_\varphi)$ which contain the identity when restricted to $\pi_0(Z(G)^{\Gamma_F})$. The first notation is better if one wants to generalize the conjecture to the case of non-quasi-split groups.

2.1 The conjecture for tori

In this section we study the conjecture in the case in which $G = T$ is a torus. We start by giving some examples of disconnected quasi-split tori. We have already seen that an example is the normalizer of the torus T of diagonal matrices in GL_n .

Example 2.4. Every action of a finite group A on a n-dimensional split torus T corresponds to an action of A on the lattice of character $X^*(T)$, and vice versa. Therefore to find actions of finite groups on T, we can look for finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$. For example, for $n = 2$, we can consider the map on $\mathbb{Z}^2 \cong \mathbb{Z}[\omega]$ with $[\omega]$ a root of $x^2 + x + 1$, given by multiplication by $[\omega]$. This map sends $1 \mapsto \omega$ and $\omega \mapsto -1 - \omega$, so it is given by the matrix $x = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. This is a matrix of order 3, and we have a map from $\mathbb{Z}/(3) \to \mathrm{GL}_n(\mathbb{Z})$ with image the subgroup generated by x. This map corresponds to the action of $\mathbb{Z}/(3)$ on $\mathbb{G}_m^2$ given by

$$(t_1, t_2) \mapsto \begin{pmatrix} t_2 & 0 \\ 0 & t_1^{-1} t_2^{-1} \end{pmatrix}.$$

We give now an example of a quasi-split disconnected torus that is not split. We consider the quasi-split torus $T'/\mathbb{Q}_p$ defined as follows: for every $\mathbb{Q}_p$-algebra R, we have

$$T'(R) := \{\left(\begin{smallmatrix} a & b \\ pb & a \end{smallmatrix}\right) \mid a, b \in R\}.$$

This is a torus since, after fixing an algebraic closure $\overline{\mathbb{Q}_p}$, we have an isomorphism $T'(\overline{\mathbb{Q}_p}) \to \mathbb{G}_m^2(\overline{\mathbb{Q}_p})$ given by sending $\left(\begin{smallmatrix} a & b \\ pb & a \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a+b\sqrt{p} & 0 \\ 0 & a-b\sqrt{p} \end{smallmatrix}\right)$. But T' is not split, since $T'(\mathbb{Q}_p) \cong (\mathbb{Q}_p[\sqrt{p}])^\times$ with isomorphism given by sending $\left(\begin{smallmatrix} a & b \\ pb & a \end{smallmatrix}\right) \mapsto a + b\sqrt{p}$.

We have two natural automorphisms on $(\mathbb{Q}_p[\sqrt{p}])^\times$ consisting of taking the inverse or acting via the Galois group. In both cases, we get an action of $\mathbb{Z}/(2)$ on T'.

In [Kal22, Section 8] the correspondence for tori is studied in the general setting of inner forms of $T \rtimes A$. Since these inner forms might not be quasi-split, the proof in this general setting is rather complicated. This is why we present an explicit description of the correspondence for the much easier case of split tori. In this case, everything can be understood using just some Clifford theory. We are going to use some facts from [Ser77]. Here Serre works with finite groups, but all his arguments extend to our case straightforwardly.

Let $\widetilde{T} \cong T \rtimes A$ be a split disconnected torus with A finite and T an n-dimensional split torus, and let $\widetilde{\pi}$ be a smooth irreducible T-tempered representation of $\widetilde{T}(F)$. From [Ser77, Section 8.2, Proposition 25], we know that $\widetilde{\pi}$ is of the form

$$\mathrm{Ind}_{T(F) \rtimes A_\pi}^{\widetilde{T}(F)} (\pi \boxtimes \rho),$$

with π an irreducible representation of $T(F)$, $A_\pi := \{a \in A \mid \pi(a \cdot t) = \pi(t)\}$ and $\rho \in \mathrm{Irr}(A_\pi)$. Moreover, every representation of this form is going to be an irreducible representation of $\widetilde{T}(F)$. We are going to use the notation $\widetilde{T}(F)_\pi := T(F) \rtimes A_\pi$. We want to understand the restriction of $\widetilde{\pi}$ to $T(F)$.

Using a classical result from Clifford theory [Ser77, Section 7.3, Proposition 22] we get that

$$\mathrm{Res}_{\widetilde{T}(F)_\pi}^{\widetilde{T}(F)} \widetilde{\pi} = \bigoplus_{a \in \widetilde{T}(F)_\pi \backslash \widetilde{T}(F) / \widetilde{T}(F)_\pi} (\pi \boxtimes \rho)^a,$$

with $(\pi \boxtimes \rho)^a(x) = (\pi \boxtimes \rho)(axa^{-1})$, for every $x \in T(F)_\pi$.

We define $\widetilde{\mathrm{LL}}_{\widetilde{T}}$ to be the map that sends $\widetilde{\pi}$ to φ_π. Notice that if $\widetilde{\pi}$ is smooth/admissible/tempered, so is π, and so is π^a for every $a \in \widetilde{T}(F)_\pi \backslash \widetilde{T}(F) / \widetilde{T}(F)_\pi$.

First, we will check that φ_{π^a} and φ_π are conjugate under A for every $a \in \widetilde{T}(F)_\pi \backslash \widetilde{T}(F) / \widetilde{T}(F)_\pi$, and therefore they correspond to the same element in $\widetilde{\Phi}(\widetilde{G})$. We look at the bijection for split connected tori that we have already discussed:

$$\mathrm{Hom}(T(F), \mathbb{C}^\times) \cong \mathrm{Hom}(F^\times \otimes X_*(T), \mathbb{C}^\times) \cong \mathrm{Hom}(F^\times, \mathbb{C}^\times \otimes X_*(\widehat{T})).$$

If π is sent to some map $f \in \mathrm{Hom}(F^\times \otimes X_*(T), \mathbb{C}^\times)$ under the first isomorphism, then π^a is going to be sent to the map f^a defined by sending $x \otimes (s_1, \dots, s_n) \mapsto f\left(x \otimes a^{-1} \cdot (s_1, \dots, s_n)\right)$, for every $x \in F^\times, (s_1, \dots, s_n) \in X_*(T)$. Now we want to apply the second isomorphism to f and f^a. If f is sent to some map $f' \in \mathrm{Hom}(F^\times, X_*(\widehat{T}) \otimes \mathbb{C}^\times)$, then f^a is sent to $a^{-1} \cdot f'$. But this is exactly how we have defined the action of a^{-1} on $\widehat{T}(\mathbb{C})$. Hence, $\varphi_\pi = a^{-1} \cdot \varphi_{\pi^a}$.

So the map that sends $\widetilde{\pi}$ to an L-parameter associated to an irreducible component of its restriction to $T(F)$ is well defined up to $\widehat{G}(\mathbb{C}) \rtimes A$-conjugacy. Defining $\widetilde{\mathrm{LL}}_{\widetilde{T}}$ like this, we have a canonical bijection between Π_{φ_π} and $\mathrm{Irr}(A_\pi)$.

Now we want to compute S_{φ_π}. We know that $T(F)$ centralizes $\mathrm{Im}(\varphi_\pi)$, so $T(F) \subset S_{\varphi_\pi}$. Moreover, the elements of A that centralize S_{φ_π} are exactly the ones in A_π. Therefore we get that $\widetilde{S_{\varphi_\pi}} \cong T(F) \rtimes A_\pi$. So we have another canonical bijection $\mathrm{Irr}(\pi_0(\widetilde{S_{\varphi_\pi}})) \cong \mathrm{Irr}(A_\pi)$, and we get a canonical bijection

$$\Pi_{\varphi_\pi} \cong \mathrm{Irr}(\pi_0(\widetilde{S_{\varphi_\pi}})).$$

This defines a map $\widetilde{\mathrm{LL}}_{\widetilde{T}}$ and finishes the proof of the conjecture in the case of split tori.

Remark 2.5. Notice that we have a canonical isomorphism $\pi_0(\widetilde{S_{\varphi_\pi}}) \cong A_\pi$ coming from the sequence 3 since $A_\pi = A^{\varphi_\pi}$ and the L-packets for connected tori are empty.

Remark 2.6. If T is just quasi-split, a similar argument can be made: instead of following the explicit map LL_T, we can just use its functoriality to conclude. Functoriality of LL_T can be found in [Yu09, Section 7.5].

Remark 2.7. Everything we said for tori is just a particular case of the more general proof of Kaletha in [Kal22, Section 8] for the case of inner forms of quasi-split tori.

3 The disconnected unramified local Langlands correspondence

In this chapter, we will define a correspondence in the case of unramified representations of disconnected reductive groups. From now on, we will assume $\widetilde{G}$ is unramified. By this, we mean that G is unramified.

Definition 3.1. We call a representation $(\widetilde{\pi}, V)$ of $\widetilde{G}(F)$ **unramified** if $V^K \neq 0$ for some hyperspecial maximal compact subgroup K of $G(F)$ or, equivalently, if its restriction to $G(F)$ contains an unramified representation. We denote the set of irreducible unramified representations of $\widetilde{G}(F)$ by $\Pi_{\mathrm{ur}}(\widetilde{G}(F))$. Analogously, we can define the notion of K-spherical representations of $\widetilde{G}(F)$.

As in the connected case, if K is a hyperspecial maximal compact subgroup of $G(F)$, then an irreducible K-spherical representation of $\widetilde{G}(F)$ is the same as a simple $\mathcal{H}(\widetilde{G}(F), K)$-module.

Remark 3.2 (Unramified correspondence for tori). Notice that in the case of tori, the map $\widetilde{\mathrm{LL}}_{\widetilde{T}}$ restricts to a correspondence between unramified representations and unramified L-parameters.

Definition 3.3. Let $\widetilde{\pi}$ be an unramified representation of $\widetilde{G}(F)$. We denote by π the unramified representation contained in $\widetilde{\pi}|_{G(F)}$. Notice that π is only defined up to precomposition with conjugation with an element of A. We will denote with ${}^a\pi$ the representation defined by ${}^a\pi(x) = \pi(axa^{-1})$. We define the map

$$\widetilde{\mathrm{LL}}_{\mathrm{ur}} : \Pi_{\mathrm{ur}}(\widetilde{G}) \to \Phi_{\mathrm{ur}}(\widetilde{G})$$

that takes $\widetilde{\pi}$ and sends it to φ_π. Here $\Phi_{\mathrm{ur}}(\widetilde{G})$ is the set of unramified L-parameter of G up to $\widehat{G} \rtimes A$-conjugacion. Notice that this map is well defined since φ_π and φ_{π^a} are equivalent up to $\widehat{G} \rtimes A$-conjugacy.

From now on, we fix K a hyperspecial maximal compact subgroup of $G(F)$ and we assume that K is the stabilizer of a point $x \in \mathcal{A}(S)$ with S the maximal split torus in the A-fixed torus T. If $a \in A$, we denote by ${}^aK$ the hyperspecial maximal compact subgroup obtained aKa^{-1}.

Remark 3.4. We have a **disconnected Cartan decomposition** of $\widetilde{G}(F)$ given by

$$\widetilde{G}(F) \cong \bigsqcup_{\substack{a \in A, \\ w \in W_K \backslash \widetilde{W} / W_K}} Kwa\,{}^{a^{-1}}K,$$

where W_K is the type of K, as in Theorem 1.19.

Notice that this Cartan decomposition becomes particularly nice in the case in which K is A-fixed. We just get

$$\widetilde{G}(F) = \bigsqcup_{\substack{a \in A, \\ w \in W_K \backslash \widetilde{W} / W_K}} KwaK.$$

This means that we have an easy base of the Hecke algebra $\mathcal{H}(\widetilde{G}(F), K)$ given by the characteristic functions $\chi_{wa} := \chi_{KwaK}$.

Proposition 3.5. *Assume K is A-fixed. Then we have an isomorphism of $\mathbb{C}$-algebras*

$$\xi_G : \mathcal{H}(G(F), K) \widetilde{\otimes} \mathbb{C}[A] \xrightarrow{\sim} \mathcal{H}(\widetilde{G}(F), K).$$

Here $\widetilde{\otimes}$ is what is called a ***crossed product*** *in [MW98]. The product structure is defined as follows: for every $a, a' \in A$ and $\chi_u, \chi_v \in \mathcal{H}(G(F), K)$,*

$$(\chi_u \otimes a)(\chi_v \otimes a') = (\chi_u * \chi_{ava^{-1}} \otimes aa').$$

The idea for this lemma comes from [Iwa66, Section 5, first proposition]. Iwahori works with generalized Tits system but his statement and ours can be proved similarly, using the disconnected Cartan decomposition instead of the Bruhat decomposition for generalized Tits system. We point out that in his paper Iwahori does not actually prove the statement. So this proof can be translated in a proof for his statement as well.

Proof. We define the map $\mathcal{H}(G(F), K) \widetilde{\otimes} \mathbb{C}[A] \xrightarrow{\xi_G} H(\widetilde{G}(F), K)$ as $\chi_u \otimes a \mapsto \chi_{ua}$. We want to prove that this is an isomorphism of algebras. It is surjective and injective thanks to the disconnected Cartan decomposition. In particular, the injectivity comes from the fact that the union is disjoint. What we need to prove is that the map is an algebra homomorphism.

Pick $\chi_u \otimes a$ and $\chi_v \otimes a'$ two elements in $\mathcal{H}(G(F), K) \widetilde{\otimes} \mathbb{C}[A]$. We denote by $m^w_{u,v}$ the number of cosets Kx contained in $Ku^{-1}Kw \cap KvK$.

Then

$$\begin{aligned} \xi_G\big((\chi_u \otimes a)(\chi_v \otimes a')\big) &= \xi_G(\chi_u * \chi_{ava^{-1}} \otimes aa') = \xi_G\Big(\sum_{w \in W_K \backslash \widetilde{W}/W_K} m^w_{u,ava^{-1}} \chi_w \otimes aa'\Big) \\ &= \sum_{w \in W_K \backslash \widetilde{W}/W_K} m^w_{u,ava^{-1}} \chi_{waa'}. \end{aligned}$$

On the other hand, the convolution

$$\xi_G(\chi_u \otimes a) * \xi_G(\chi_v \otimes a') = \chi_{ua} * \chi_{va'} = \sum_{\substack{b \in A, \\ w \in W_K \backslash \widetilde{W}/W_K}} m^{wb}_{ua,va'} \chi_{wb}.$$

First, we want to prove that if $b \neq aa'$ then $m^{wb}_{ua,va'} = 0$. We need to count the cosets of the form Kx in the intersection $Ka^{-1}u^{-1}Kw \cap Kva'b^{-1}K$. If $b \neq aa'$, then $Ka^{-1}u^{-1}Kw$ is contained in the coset $G(F)a^{-1}$ of $\widetilde{G}(F)$, and $Kva'b^{-1}K$ is contained in the coset $G(F)a'b^{-1}$. Therefore the intersection must be empty and $m^{wb}_{ua,va'} = 0$.

To conclude the proof, we need to show that if $b = aa'$ then $m^{wb}_{ua,va'} = m^w_{u,ava^{-1}}$. Let C denote the set of cosets of the form Kx contained in $Ka^{-1}u^{-1}Kwaa' \cap Kva'K \subset G(F)$ and let D the set of cosets Ky contained in $Ku^{-1}Kw \cap Kava^{-1}K$. Then the map from C to D that sends Kx to $Kaxaa'$ is a bijection between C and D, proving the desired statement. $\square$

Remark 3.6 (Disconnected Satake isomorphism). This isomorphism gives us a link between K-spherical representations and unramified representations of a torus, as in the connected case. The map

$$\widetilde{\mathscr{S}} = \mathscr{S} \otimes \mathrm{Id} : \mathcal{H}(G(F)(F), K) \widetilde{\otimes} \mathbb{C}[A] \to \mathcal{H}(T(F)(F), T(F) \cap K)^W \widetilde{\otimes} \mathbb{C}[A]$$

is an algebra isomorphism.

We just need to show that it is an algebra homomorphism. Let $\chi_u \otimes a, \chi_v \otimes a' \in \mathcal{H}(G(F), K) \mathbin{\widetilde{\otimes}} \mathbb{C}[A]$. Then

$$\mathscr{S}\big((\chi_u \otimes a)(\chi_v \otimes a')\big) = \mathscr{S}(\chi_u * \chi_{ava^{-1}} \otimes aa') = \big(\sum_{w \in W_K \backslash \widetilde{W}/W_K} c(w,u) b_w\big) * \big(\sum_{w \in W_K \backslash \widetilde{W}/W_K} c(w, ava^{-1}) b_w\big) \otimes aa',$$

where c is the same as in the proof of Theorem 1.20. On the other hand

$$\mathscr{S}(\chi_u \otimes a)\mathscr{S}(\chi_v \otimes a') = \big(\sum_{w \in W_K \backslash \widetilde{W}/W_K} c(w,u) b_w\big) * \big(\sum_{w \in W_K \backslash \widetilde{W}/W_K} c(w,v) b_{awa^{-1}}\big) \otimes aa'.$$

Therefore, if $\sum_{w \in W_K \backslash \widetilde{W}/W_K} c(w,v) b_{awa^{-1}} = \sum_{w \in W_K \backslash \widetilde{W}/W_K} c(w, ava^{-1}) b_w$ we are done. This is true since $c(w,u) = \mu(KuK \cap U(F)wK)$ and since the unipotent radical U of B is fixed under the action of A.

Remark 3.7. Notice that the assumption of K to be A-fixed is not as restrictive as it might seem. In fact, $G(F)$ must have an A-fixed hyperspecial maximal compact subgroup K corresponding to a vertex in $\mathcal{A}(S)$. This is true since A acts via pinned automorphisms.

Remark 3.8. Notice that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{H}(\widetilde{G}(F), K) \arrow[rr, "\xi_G"] & & \mathcal{H}(G(F), K) \mathbin{\widetilde{\otimes}} \mathbb{C}[A] \\
& \mathcal{H}(G(F), K) \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

with diagonal arrows given by the obvious inclusions. Therefore for studying irreducible K-spherical representation of $\widetilde{G}(F)$, we can just study simple $\mathcal{H}(G(F), K) \mathbin{\widetilde{\otimes}} \mathbb{C}[A]$-modules whose restriction to $\mathcal{H}(G(F), K)$ contains an unramified representation.

Theorem 3.9 (Theorem 1.3.[MW98])**.** *Let π be a simple $\mathcal{H}(G(F), K)$-module, A_π its stabilizer in A. There is an equivalence between the category of $\mathbb{C}[A_\pi]$-modules and the category of those $\mathcal{H}(G(F), K) \mathbin{\widetilde{\otimes}} \mathbb{C}[A]$-modules whose restriction to $\mathcal{H}(G(F), K)$ is isomorphic to a direct sum of copies of conjugates of π. This equivalence is defined by*

$$\rho \mapsto \big(\mathcal{H}(G(F), K) \mathbin{\widetilde{\otimes}} \mathbb{C}[A]\big) \otimes_{\mathcal{H}(G(F),K) \mathbin{\widetilde{\otimes}} \mathbb{C}[A_\pi]} (\pi \otimes \rho) =: \mathrm{Ind}_{\mathcal{H}(G(F),K) \mathbin{\widetilde{\otimes}} \mathbb{C}[A_\pi]}^{\mathcal{H}(G(F),K) \mathbin{\widetilde{\otimes}} \mathbb{C}[A]} (\pi \otimes \rho)$$

for every simple $\mathbb{C}[A_\pi]$-module ρ.

Remark 3.10. All of this can be translated in terms of representation of groups, obtaining the same result that we would have obtained using the classification of representation of quasi-split tori and Remark 3.6. Notice that in this theorem we are implicitly using that $\mathcal{H}(G, K)$ is commutative in order to give $\pi \otimes \rho$ a canonical structure of $\mathcal{H}(G(F), K) \mathbin{\widetilde{\otimes}} \mathbb{C}[A_\pi]$-module.

Remark 3.11. In [MW98] they work with the crossed product of a finite dimensional algebra H with the group algebra of a finite group A, but they never use that H is finite dimensional to prove this theorem. They just reference [Dad86] and in this paper H is not required to be finite-dimensional. In particular, this theorem is a corollary of [Dad86, Theorem 10.6, Corollary 11.16].

Theorem 3.12. *Assume that G is adjoint, let $\varphi \in \Phi_{\mathrm{ur}}(\widetilde{G})$ and let K be an A-fixed hyperspecial maximal compact subgroup. If there is a bijection $\Pi_\varphi \cong \mathrm{Irr}(\pi_0(S_\varphi), \mathrm{Id})$, then the map $\widetilde{\mathrm{LL}}_{\mathrm{ur}}$ defines a map from $\Pi_{\mathrm{ur}}(\widetilde{G})$ to the set $\Phi_{\mathrm{ur}}(\widetilde{G})$, and $\widetilde{\Pi}_\varphi$ is in bijection with $\mathrm{Irr}(\pi_0(\widetilde{S_\varphi}))$.*

Proof. Notice again that we can always find an A-fixed hyperspecial maximal compact subgroup (Remark 3.7). Since all the hyperspecial maximal subgroups of $G(F)$ are conjugate the set of unramified representations of $\widetilde{G}(F)$ is just the set of spherical representations with respect to a fixed hyperspecial maximal compact subgroup. We study the K-spherical representations. Let now $\widetilde{\pi}$ be a K-spherical representation of $\widetilde{G}(F)$ such that the restriction to $\mathcal{H}(G(F), K)$ contains an irreducible unramified representation π. Applying ξ_G to π we get a simple $\mathcal{H}(G(F), K) \widetilde{\otimes} \mathbb{C}[A]$-module. From Theorem 3.9 we have that the fiber $\widetilde{\mathrm{LL}}_{\mathrm{ur}}^{-1}(\varphi_\pi)$ is in bijection with $\mathrm{Irr}(A_\pi)$.

We recall now that we have a short exact sequence

$$1 \to \pi_0(S_{\varphi_\pi}) \to \pi_0(\widetilde{S_{\varphi_\pi}}) \to A^{\varphi_\pi} \to 1,$$

and we notice that $A^{\varphi_\pi} = A_\pi$. Since G is adjoint $\pi_0(S_{\varphi_\pi})$ is trivial, therefore $\mathrm{Irr}(\pi_0(\widetilde{S_\varphi}))$ is in bijection with $\mathrm{Irr}(A^{\varphi_\pi}) = \mathrm{Irr}(A_\pi)$. Therefore $\widetilde{\mathrm{LL}}_{\mathrm{ur}}$ induces a bijection

$$\widetilde{\Pi}_\varphi \to \mathrm{Irr}(\pi_0(\widetilde{S_\varphi})). \qquad \square$$

Remark 3.13. Notice that the only thing that we use in this proof is that $\pi_0(S_\varphi)$ is trivial and that all the hyperspecial maximal compact subgroups are conjugate. Therefore the theorem applies to $G = \mathrm{GL}_n$ as well.

Example 3.14. We compute one very easy explicit example. Let $G = \mathrm{GL}_2 / \mathbb{Q}_p$ and let $A = \langle \sigma \rangle$ be a group of order 2 acting on G via $\sigma \cdot x = \frac{x}{\det(x)}$. Let φ an unramified L-parameter with $\varphi(\mathrm{frob}) = (a, b) \in \widehat{T}(\mathbb{C})$ with T the torus of diagonal matrices in GL_2. Then $\langle \sigma \rangle^\varphi$ is trivial if $a \neq b^{-1}$ and it is everything otherwise. Now we move to the representation theory side. Since the following diagram commutes

$$\begin{tikzcd}
\mathcal{H}(G(F), K) \widetilde{\otimes} \mathbb{C}[A] \arrow[r, "\widetilde{\mathscr{S}}"] & \mathcal{H}(T(F), T(F) \cap K)^W \widetilde{\otimes} \mathbb{C}[A] \\
\mathcal{H}(G(F), K) \arrow[u, hook] \arrow[r, "\mathscr{S}"'] & \mathcal{H}(T(F), T(F) \cap K)^W \arrow[u, hook]
\end{tikzcd}$$

it is sufficient to work with representation of the torus T. Then

$$\chi_\varphi(t_1, t_2) = a^{\mathrm{val}(t_1)} b^{\mathrm{val}(t_2)}.$$

We are know looking for representations of $T(F) \rtimes \langle \sigma \rangle$ that contains χ_φ when restricted to $T(F)$. So we just need to compute $\langle \sigma \rangle_{\chi_\varphi}$. Then

$$\sigma \chi_\varphi(t_1, t_2) = \chi_\varphi(t_2^{-1}, t_1^{-1}) = a^{-\mathrm{val}(t_2)} b^{-\mathrm{val}(t_1)}.$$

For this to be equal to $a^{\mathrm{val}(t_1)} b^{\mathrm{val}(t_2)}$ for every $t_1, t_2 \in \mathbb{Q}_p$, we necessary need $a = b^{-1}$. Therefore we get a bijection $\widetilde{\Pi}_\varphi \to \mathrm{Irr}(\widetilde{S}_\varphi)$ as desired.

3.1 Non-fixed hyperspecial subgroup

Now we study the case in which the hyperspecial maximal compact subgroups are not A-fixed. We start by understanding what happens in the connected case.

Let K, K' be two hyperspecial maximal compact subgroups of $G(F)$, that are not conjugate. For simplicity, we assume that K and K' are stabilizers of two points in $\mathcal{A}(S)$. Let $G_{\mathrm{ad}} := G/Z(G)$ be the adjoint group of G and let $K_{\mathrm{ad}}, K'_{\mathrm{ad}}$ be the images of K and K' respectively in $G_{\mathrm{ad}}(F)$. These are hyperspecial maximal compact subgroups of $G_{\mathrm{ad}}(F)$, therefore there exists $h \in G_{\mathrm{ad}}(F)$ such that $hKh^{-1} = K'$.

Proposition 3.15. *Such an element h leads to an isomorphism*

$$\mathcal{H}(G_{\mathrm{ad}}, K_{\mathrm{ad}}) \xrightarrow{h^*} \mathcal{H}(G_{\mathrm{ad}}, K'_{\mathrm{ad}})$$

given by $f(-) \mapsto f\left(h^{-1}(-)h\right)$.

Proof. The only thing that we need to check is that h^* is an algebra homomorphism. We pick two elements $\chi_{KuK}, \chi_{KvK} \in \mathcal{H}(G_{\mathrm{ad}}, K_{\mathrm{ad}})$, and notice that $h^*(\chi_{KuK}) = \chi_{K'h^{-1}uhK'}$. Then

$$h^*\big(\chi_{KuK} * \chi_{KvK}\big)(x) = \int_{KuK} \chi_{KvK}(g^{-1}hxh^{-1})\, dg = \mu(KuK \cap hxh^{-1}Kv^{-1}K)$$

for every $x \in G_{\mathrm{ad}}(F)$. On the other hand

$$h^*(\chi_{KuK}) * h^*(\chi_{KvK})(x) = \int_{G(F)} h^*(\chi_{KuK})(g)h^*(\chi_{KvK})(g^{-1}x)\, dg = \mu(h^{-1}KuKh \cap xh^{-1}Kv^{-1}Kh).$$

The measure of these two sets is the same since one is obtained from the other by conjugating with h^{-1} and $G_{\mathrm{ad}}(F)$ is unimodular. So h^* is an algebra homomorphism. $\square$

We want to "transport" this isomorphism to G. The map $G(F) \to G_{\mathrm{ad}}(F)$ is not surjective in general, but the action of G on itself by conjugation, factors through an action of G_{ad} on G. Therefore $G_{\mathrm{ad}}(F)$ acts on $G(F)$. We denote the action of h on an element $x \in G(F)$ by by $h \cdot x$. Since K and K' are not conjugate, $h \notin \mathrm{Im}\big(G(F) \to G_{\mathrm{ad}}(F)\big)$, but we still have that $h \cdot K = K'$.

Proposition 3.16. *Such an element h induces again an isomorphism*

$$\mathcal{H}(G(F), K) \xrightarrow{h^*} \mathcal{H}(G(F), K')$$

given by $f(-) \mapsto f\left(h \cdot (-)\right)$.

Proof. We check that this is an algebra homomorphism: doing similar computations as in the proof of Proposition 3.15, it is sufficient to show that $\mu(h \cdot X) = \mu(X)$ for every measurable subset $X \subset G(F)$. First, we notice that the measure ${}^h\mu$ given by ${}^h\mu(X) = \mu(h \cdot X)$ is a Haar measure. In fact, if $X \subset G(F)$ and $g \in G(F)$, then

$${}^h\mu(gX) = \mu(h \cdot (gX)) = \mu((h \cdot g)(h \cdot X)) = \mu(h \cdot X).$$

Therefore, ${}^h\mu = c\mu$ for some real number c. The proposition follows from the following claim:

Claim: There exists $n \in \mathbb{N}$ such that $h^n \in G(F)/Z(G(F))$.

If the claim is true, then ${}^{h^n}\mu = \mu$ and so $c = 1$, and ${}^h\mu = \mu$. We proceed to prove the claim: we first assume that G is semisimple and we consider the exact sequence

$$1 \to Z(G)(F) \to G(F) \to G_{\mathrm{ad}}(F) \xrightarrow{\delta} H^1(F, Z(G)(F)) \to \cdots .$$

Since G is semisimple, $Z(G)$ is finite. Therefore, $H^1(F, Z(G)(F))$ is torsion [Mil20, Chapter 2, Corollary 4.3]. So, if $h \in G_{\mathrm{ad}}(F))$, then $\delta(h^n) = 0$ for some $n \in \mathbb{N}$ meaning that $h^n \in \mathrm{Im}(G(F) \to G_{\mathrm{ad}}(F))$.

To conclude, if G is not semisimple, we can consider the same exact sequence using $G_{\mathrm{der}} = [G, G]$ instead of G, getting that for some $n \in \mathbb{N}$, $h^n \in G_{\mathrm{der}}(F)/Z(G_{\mathrm{der}})(F) \subset G(F)/Z(G)(F)$. $\square$

Remark 3.17. Notice that after a suitable choice of h (h not in the image of the isogeny $G(F) \to G_{\mathrm{ad}}(F)$), the isomorphism h_G^* is compatible with $\mathrm{LL}_{\mathrm{ur}}$. By this, we mean that if π is an $\mathcal{H}(G, K)$-module then $\varphi_\pi = \varphi_{h^*\pi}$ up to $\widehat{G}$-conjugation.

We choose h to be an element in $G_{\mathrm{ad}}(F)$ that conjugates K_{ad} and K'_{ad} and fixes $\mathcal{A}(S)$, so that $h \in N_{G_{\mathrm{ad}}}(S_{\mathrm{ad}})(F)$. We can always find such an h since, if $\mathcal{I}$ is the F-points of the Iwahori subgroup associated to our A-fixed pinning, then

$$\mathcal{I}W_{K'}\mathcal{I} = \mathcal{I}h\mathcal{I}W_K\mathcal{I}h^{-1}\mathcal{I},$$

so $h \in \mathcal{I}\backslash G_{\mathrm{ad}}(F)/\mathcal{I}$. From [KP23, Theorem 7.8.1] we have that the inclusion $N_{G_{\mathrm{ad}}}(S_{\mathrm{ad}}) \hookrightarrow G_{\mathrm{ad}}$ induces a bijection

$$\mathcal{I}\backslash G_{\mathrm{ad}}(F)/\mathcal{I} \longleftrightarrow N_{G_{\mathrm{ad}}}(S_{\mathrm{ad}})(F)/\big(T_{\mathrm{ad}}(F)\big)^0.$$

Therefore, h has a representative in $\mathcal{I}\backslash G_{\mathrm{ad}}(F)/\mathcal{I}$ that lies in $N_{G_{\mathrm{ad}}}(S_{\mathrm{ad}})(F)$.

After this choice, h induces another isomorphism $h_T^* : \mathcal{H}(T(F), T(F)\cap K)^W \to \mathcal{H}(T(F), T(F)\cap K')^W$ that fits into the commutative diagram

$$\begin{tikzcd}
\mathcal{H}(G(F), K) \arrow[r, "\mathscr{S}"] \arrow[d, "h^*"] & \mathcal{H}(T(F), T(F)\cap K)^W \arrow[d, "h_T^*"] \\
\mathcal{H}(G(F), K') \arrow[r, "\mathscr{S}"] & \mathcal{H}(T(F), T(F)\cap K')^W .
\end{tikzcd}$$

Clearly, $\mathcal{H}(T(F), T(F)\cap K)^W$ is equal to $\mathcal{H}(T(F), T(F)\cap K')^W$ since there is only one maximal compact subgroup in $T(F)$. So h_T^* is just twisting $\mathcal{H}(T(F), T(F)^1)^W$ by some element in $(N_{G_{\mathrm{ad}}}(S_{\mathrm{ad}}))(F)$. If $h \in S_{\mathrm{ad}}(F)$, then its action on $T(F)$ is trivial. So we can assume $h \in N_{G_{\mathrm{ad}}}(S_{\mathrm{ad}})/S_{\mathrm{ad}}(F)$. But this is exactly the relative Weyl group W of G, since the canonical surjection $G \to G_{\mathrm{ad}}$ induces an isomorphism $N_{G_{\mathrm{ad}}}(S_{\mathrm{ad}})/S_{\mathrm{ad}} \cong N_G(S)/S$. This means that h_T^* is just the identity map on $\mathcal{H}(T(F), T(F)^1)^W$. Therefore $\varphi_\pi = \varphi_{h^*\pi}$.

Proposition 3.15, Proposition 3.16 and Remark 3.17 show us that is not necessary to "understand" the Hecke algebras $\mathcal{H}(G(F), K)$ for every hyperspecial maximal compact subgroup K in order to understand unramified representations of $G(F)$. It is sufficient to understand one of them and to have a family of well-behaved isomorphism between the different Hecke algebras. Since in the disconnected case we understand just one Hecke algebra, we would like to apply this approach to $\widetilde{G}$.

Let K be again an A-fixed hyperspecial maximal compact subgroup of $G(F)$, and let $\{K_i\}_{i\in I}$ be the set of hyperspecial maximal compact subgroups up to conjugacion. The first problem that we encounter is that in general G_{ad} does not act on $\widetilde{G}$, therefore we do not have an isomorphism $\widetilde{h_G^*}$ that makes the following diagram commute

$$\begin{tikzcd}
\mathcal{H}(\widetilde{G}(F), K) \arrow[r, dashed, "\widetilde{h^*}"] & \mathcal{H}(\widetilde{G}(F), K') \\
\mathcal{H}(G(F), K) \arrow[r, "h^*"] \arrow[u, hook] & \mathcal{H}(G(F), K') \arrow[u, hook]
\end{tikzcd} \tag{4}$$

Assumption 1: From now on we will assume that the action of A on $Z(G)$ is trivial.

We first look at what this assumption implies on the automorphic side. Since A fixes the center point-wise, G_{ad} acts on $\widetilde{G}$ and an isomorphism h^* like the one in Proposition 3.16 lifts to an isomorphism $\widetilde{h^*}$ as in the diagram 4.

For every K_i hyperspecial maximal compact subgroup of $G(F)$ that is not A-fixed, we fix an element $h_i \in G_{\mathrm{ad}}(F)$ that leads to isomorphisms

$$h_i^* : \mathcal{H}(G(F), K) \to \mathcal{H}(G(F), K_i)$$

and

$$\widetilde{h_i^*} : \mathcal{H}(\widetilde{G}(F), K) \to \mathcal{H}(\widetilde{G}(F), K_i).$$

Let π be an irreducible, non-trivial $\mathcal{H}(G(F), K)$-module and let φ_π be the associated L-parameter. Applying h_i^* for some $i \in I$, we get an $\mathcal{H}(G(F), K_i)$-module $h_i^*\pi$, and doing this for every $i \in I$ we get the entire Π_{φ_π} by Remark 3.17. Therefore, in order to understand the set $\widetilde{\Pi}_{\varphi_\pi}$ we need to understand how many $\mathcal{H}(\widetilde{G}(F), K_i)$-modules contain $h_i^*\pi$ when restricted to $\mathcal{H}(G(F), K_i)$ for every $i \in I$. Using the commutative diagram 4, we know that this is exactly the number of $\mathcal{H}(\widetilde{G}(F), K)$-modules that contain π when restricted to $\mathcal{H}(G(F), K)$, and these are in bijection with $\mathrm{Irr}(A_\pi)$ by Theorem 3.9.

Notice that this method might not give a full understanding of $\widetilde{\Pi}_{\varphi_\pi}$ without being checked further. In fact, it does not take into account that if $a \in A \setminus A_\pi$, then ${}^a\pi$ does not need to belong to Π_{φ_π}, but an irreducible representation of $\widetilde{G}(F)$ which restricted to $G(F)$ contains ${}^a\pi$ will be in $\widetilde{\Pi}_{\varphi_\pi}$. But this is not a problem since if the restriction to $G(F)$ of a representation of $\widetilde{G}(F)$ contains π then it will contain ${}^a\pi$ as well. On the other hand, if $h_i^*\pi \cong {}^a h_j^*\pi$ for some $a \in A_\pi, i \neq j \in I$, then if the restriction to $G(F)$ of an irreducible representation of $\widetilde{G}(F)$ contains $h_i^*\pi$, it will not contain $h_j^*\pi$. This follows again from the description given in [Ser77, Section 7.3, Proposition 22] for restrictions of representations of the semidirect product of two groups. Therefore, $\widetilde{\Pi}_{\varphi_\pi}$ is in bijection with pairs (π', ρ) with $\pi' \in \Pi_{\varphi_\pi}$ and $\rho \in \mathrm{Irr}(A_\pi)$.

We now check what happens on the Galois side. Let φ an unramified L-parameter. Then the group $\pi_0(S_\varphi)$ is abelian since the image of φ is determined by $\varphi(\mathrm{frob}) = (s_0, \mathrm{frob}) \in {}^L G$, and the centralizer of such an element is always abelian [Ste68, Lemma 9.2]. But we do not know if the exact sequence

$$1 \to \pi_0(S_\varphi) \to \pi_0(\widetilde{S_\varphi}) \to A^\varphi \to 1$$

splits, therefore we can't use the description of representations of the semidirect product of group used in Section 2.1.

Let $s := \varphi(\mathrm{frob}) \in \widehat{G}(\mathbb{C})$. We are going to now add the following assumptions:

Assumption 2: From now on we will assume that G is semisimple and that $a \cdot s = s$ for every $a \in A^\varphi$.

This assumption immediately imply that the sequence

$$1 \to \pi_0(S_\varphi) \to \pi_0(\widetilde{S_\varphi}) \to A^\varphi \to 1$$

splits, and we can use [Ser77, Section 8.2, Proposition 25] to get that every irreducible representation of $\pi_0(\widetilde{S_\varphi})$ is given by an irreducible representation α of $\pi_0(S_\varphi)$ and an irreducible representation ρ of A^φ_α.

Lemma 3.18. *Under assumption 2, there is an embedding*

$$\pi_0(S_\varphi) \hookrightarrow \pi_1(\widehat{G})/(1-s)\pi_1(\widehat{G}),$$

where $\pi_1(\widehat{G})$ is the ***algebraic fundamental group*** *of $\widehat{G}$, defined as the kernel of the canonical central isogeny $\widehat{G}_{\mathrm{sc}} \to \widehat{G}$, where $\widehat{G}_{\mathrm{sc}}$ is the universal cover of $\widehat{G}$. Moreover this embedding is A-equivariant.*

Proof. First we check that A actually acts on $\pi_1(\widehat{G})$. The action of A on $\widehat{G}$ lifts to an action of $\widehat{G}_{\mathrm{sc}}$. Moreover, from the exact sequence

$$1 \to \pi_1(\widehat{G}) \to \widehat{G}_{\mathrm{sc}} \to \widehat{G} \to 1,$$

we get an exact sequence

$$1 \to \pi_1(\widehat{G}) \to \widehat{T}_{\mathrm{sc}} \to \widehat{T} \to 1,$$

where $\widehat{T}_{\mathrm{sc}}$ is the preimage of $\widehat{T}$ in $\widehat{G}_{\mathrm{sc}}$. Now we have an action of A on $X^*(\widehat{T})$ and on $X^*(\widehat{T}_{\mathrm{sc}})$, therefore we get an action on $X^*(\pi_1(\widehat{G})) = X^*(\widehat{T}_{\mathrm{sc}})/X^*(\widehat{T})$, and so an action on $\pi_1(\widehat{G})$.

The embedding is a trivial corollary of [Ste68, Lemma 9.2], and the map is given by sending an element $x \in S_\varphi$ to $\tilde{x}s\tilde{x}^{-1}s^{-1}$, where $\tilde{x} \in \widehat{G}_{\mathrm{sc}}$ and $\tilde{x} \mapsto x$ under the canonical isogeny. Then, if $x \in S_\varphi$

$$a \cdot x \mapsto (a \cdot \tilde{x})(s(a \cdot \tilde{x})s^{-1})^{-1} = (a \cdot \tilde{x})(a \cdot s(a \cdot \tilde{x}^{-1})s^{-1}) = a \cdot (\tilde{x}s\tilde{x}^{-1}s^{-1}),$$

where the last equality is using that $a \cdot s = s$. ☐

Lemma 3.19. *Under the assumptions 1 and 2, the action of A^φ on $\pi_0(S_\varphi)$ is trivial.*

Proof. Thanks to Lemma 3.18, we just need to show that the action on $\pi_1(G)$ is trivial. We define $\pi_1'(\widehat{G}) := X_*(\widehat{T})/X_*(\widehat{T}_{\mathrm{ad}})$ and we call it the **Borovoi fundamental group** of $\widehat{G}$. Then

$$\pi_1'(\widehat{G}) = X_*(\widehat{T})/X_*(\widehat{T}_{\mathrm{sc}}) = X^*(T)/X^*(T_{\mathrm{ad}}) = X^*(Z(G)).$$

We know by assumption 1 that the action of A is trivial, therefore we have a trivial action of A on $\pi_1'(G)$. The only thing that is left to understand is the relation between the Borovoi fundamental group and the algebraic fundamental group. This is studied in [Bor98, Section 1]. In particular, in [Bor98, Proposition 1.11] Borovoi proves that we have a canonical isomorphism

$$\pi_1'(\widehat{G}) \xrightarrow{\theta} \operatorname{Hom}\left(\pi_1^{\mathrm{top}}(\mathbb{G}_m(\mathbb{C})), \pi_1^{\mathrm{top}}(\widehat{G}(\mathbb{C}))\right),$$

where $\pi_1^{\mathrm{top}}(\widehat{G}(\mathbb{C}))$ and $\pi_1^{\mathrm{top}}(\mathbb{G}(\mathbb{C}))$ are the topological fundamental groups of $\widehat{G}(\mathbb{C})$ and $\mathbb{G}_m(\mathbb{C})$ respectively. We have an action on A on $\operatorname{Hom}\left(\pi_1^{\mathrm{top}}(\mathbb{G}_m(\mathbb{C})), \pi_1^{\mathrm{top}}(\widehat{G}(\mathbb{C}))\right)$ given by the action of A on $\pi_1^{\mathrm{top}}(\widehat{G}(\mathbb{C}))$, and this makes θ an A-equivariant map. To show this we write down θ explicitly. First θ maps $X_*(\widehat{T})$ to $\operatorname{Hom}\left(\pi_1^{\mathrm{top}}(\mathbb{G}_m(\mathbb{C}), \pi_1^{\mathrm{top}}(\widehat{T}(\mathbb{C}))\right)$ using the functoriality of π_1^{top} and then embeds it in $\operatorname{Hom}\left(\pi_1^{\mathrm{top}}(\mathbb{G}_m(\mathbb{C}), \pi_1^{\mathrm{top}}(\widehat{G}(\mathbb{C}))\right)$. These maps are both obviously A-equivariant, therefore θ is.

Fixing an isomorphism $\pi_1^{\mathrm{top}}(\mathbb{G}_m(\mathbb{C})) \cong \mathbb{Z}$ determines isomorphisms

$$\operatorname{Hom}\left(\pi_1^{\mathrm{top}}(\mathbb{G}_m(\mathbb{C}), \pi_1^{\mathrm{top}}(\widehat{G}(\mathbb{C}))\right) \cong \pi_1^{\mathrm{top}}(\widehat{G}(\mathbb{C})) \cong \pi_1(\widehat{G})(\mathbb{C}).$$

The second isomorphism is A-equivariant since the action of A comes from the A-equivariant exact sequence

$$1 \to \pi_1(G) \to \widehat{G}_{\mathrm{sc}} \to \widehat{G} \to 1.$$

The action of A on $\pi_1(\widehat{G})$ is determined by the action on $\pi_1(\widehat{G})(\mathbb{C})$, but we just proved that this is trivial. $\square$

Theorem 3.20. *We assume assumptions 1 and 2, and we assume that $\varphi = \varphi_\pi$ for some $\mathcal{H}(G(F), K)$-module π. Moreover, assume that Π_φ is in bijection with* $\operatorname{Irr}(\pi_0(S_\varphi), \mathrm{Id})$. *Then $\widetilde{\Pi}_\varphi$ is in bijection with the set of irreducible representations of $\widetilde{S_\varphi}$.*

Proof. Since the sequence

$$1 \to \pi_0(S_\varphi) \to \pi_0(\widetilde{S_\varphi}) \to A^\varphi \to 1$$

splits, then $\operatorname{Irr}(\pi_0(\widetilde{S_\varphi}))$ is in bijection with the pairs (α, ρ) with $\alpha \in \operatorname{Irr}(\pi_0(S_\varphi))$ and $\rho \in \operatorname{Irr}(A^\varphi_\alpha)$. But the action of A^φ on $\pi_0(S_\varphi)$ is trivial by Lemma 3.19, therefore $A^\varphi_\alpha = A^\varphi$ for every $\alpha \in \operatorname{Irr}(\pi_0(S_\varphi))$.

We recall that we have a bijection between $\widetilde{\Pi}_\varphi$ and the pairs (π', ρ) with $\pi' \in \Pi_\varphi$ and $\rho \in \operatorname{Irr}(A^\varphi)$. Therefore, the bijection $\Pi_\varphi \to \operatorname{Irr}(\pi_0(S_\varphi), \mathrm{Id})$ concludes the proof. $\square$

Remark 3.21. We conclude with a comment on the assumptions that we had in the last theorem. Everything relies on assumption 1, since it is the assumptions that makes us understand $\mathcal{H}(G, K')$-modules for a non A-fixed hyperspecial maximal compact subgroup K'. The assumption of G being semisimple is probably easy to drop, and it is mainly needed to ensure the algebraic fundamental group has nice properties. The real problem is that if the sequence

$$1 \to \pi_0(S_\varphi) \to \pi_0(\widetilde{S_\varphi}) \to A^\varphi \to 1$$

does not split, then the representations of $\pi_0(\widetilde{S_\varphi})$ are not easily accessible. One could try to drop the hypothesis of A^φ fixing $\varphi(\mathrm{frob})$, but without the splitting we would not get an easy description of these representations. We still hope that the sequence might always split, saving us from using some facts from the representation theory of abelian extension of finite groups.

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