# Kleine AG, Talk 1:
# Galois representation

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In these notes we are going to mainly follow the first pages of Toby Gee's notes on modularity lifting theorems. The point is not to write better notes (that is probably very hard) but just to have notes on what is going to happen in this talk.

For the rest of the notes, we will denote by $p > 2$ a prime bigger than 2. If $K$ is a field, we denote by $G_K$ the absolute Galois group of $K$, by $\mathcal{O}_K$ it's ring of integers and by $\varpi_K$ a uniformiser for $\mathcal{O}_K$.

After a brief introduction about the Weil group and the statement of the local class field theory, we are going to state Grothendieck's monodromy theorem for Galois representation with $p \neq l$. After that, we'll say something about the case $p = l$, giving the definition of Hodge-Tate representation. In the end we are going to say define geometric Galois representation for number fields, and state the Fontaine-Mazur conjecture.

## 1 Recollections

In this section we will just recall what is the Weil group of a local field and what is the statement of the local class field theory. Let $K$ being a finite extension of $\mathbb{Q}_l$, for some prime $l \neq p$. We start by recalling that, since the natural action of $G_K$ on $K$ preserves the valuation, this induces an action on the residue field $k$ and therefore a map $G_K \to G_k$. We get a short exact sequence

$$1 \to I_K \to G_K \to G_k \to 1$$

with $I_K$ defined as the kernel of the last map, and called the **inertia subgroup**. Thanks to this sequence we can define the Weil group to be the preimage of $Frob_k^{\mathbb{Z}}$ with $Frob_k \in G_k$ the Frobenius:

$$\begin{tikzcd}
1 \arrow[r] & I_F \arrow[r] \arrow[d, equal] & G_K \arrow[r] & G_k \arrow[r] & 1 \\
1 \arrow[r] & I_F \arrow[r] & W_K \arrow[r] \arrow[u, hook] & Frob_k^{\mathbb{Z}} \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

The Weil group is a topological group, but not with the subspace topology. In fact, we chose the topology that makes $I_K$ open.

We will denote by $K^{ur}$ the maximal unramified extension of $K$ and by $K^{tame} = \bigcup_{(m,l)=1} K^{ur}(\varpi_K^{1/m})$ the maximal tamely ramified extension[1]. Then we denote $P_K := Gal(\overline{K}/K^{tame})$ the wild inertia subgroup of $G_K$. Given now a compatible system of primitive roots of unity[2] $\zeta = (\zeta_m)_{(m,l)=1}$, we get a character

$$t_\zeta : I_K/P_K \xrightarrow{\sim} \prod_{p \neq l} \mathbb{Z}_p$$

[1] $L/K$ is tamely ramified if for any maximal ideal $\mathfrak{q}$ of $\mathcal{O}_L$ lying over $\mathfrak{p}$, we have that $l/k$ is separable and $l$ does not divide the degree of $L$ over the maximal unramified extension of $K$ in $L$.

[2] This means that $\zeta_{mn}^n = \zeta_m$.

defined by

$$\frac{\sigma(\varpi_K^{1/m})}{\varpi_K^{1/m}} = \zeta_m^{(t_\zeta(\sigma) \pmod m)}.$$

We will denote by $t_{\zeta,p}$ the composition of $t_\zeta$ with the projection on $\mathbb{Z}_p$.

The local class field theory, gives a interesting connection between the Weil group of a field, and the field itself:

**Theorem 1.1** (Local class field theory)**.** *There are unique isomorphism $Art_K : K^\times \to W_K^{ab}$ such that for every finite extension $K'/K$, we have that $Art_{K'} = Art_K \circ N_{K'/K}$ and moreover, the following diagram commutes:*

$$\begin{tikzcd} K^\times \arrow[r, "Art_K"] \arrow[d, two heads, "val_K"'] & W_K^{ab} \arrow[d, two heads] \\ \mathbb{Z} \arrow[r] & Frob_k^{\mathbb{Z}} \end{tikzcd}$$

The representation of $W_K^{ab}$, are just characters of $W_K$, and this theorem gives us a correspondence between this characters and representations of $K^\times = GL_1(K)$. We can see this as the easiest case of the local Langlands conjecture.

# 2 Introduction to Galois representation

Let now $K$ a field with a fixed separable closure $\overline{K}$, and let $L$ being any topological field. We recall that the Galois group $G_K$ has a natural profinite topology.

**Definition 2.1.** A **Galois representation** is a continuous homomorphism $\rho : G_K \to GL_n(L)$ for some $n$.

Clearly the nature of this representations strongly depends on $L$. For example, if $L$ has the discrete topology, the image of $\rho$ is finite ($G_K$ is compact) and, therefore, $\rho$ factors through a finite Galois group $Gal(K'/K)$.

On the other hand, if we consider $L$ to be a finite extension of $\mathbb{Q}_p$ there can be example of representation with infinite image.

*Example* 2.2. One of the most important examples in which the image of a Galois representation is infinite, is the $p$-adic cyclotomic character: let $L = \mathbb{Q}_p$ and $char(K) \neq 0$. Then we can define $\varepsilon_p : G_K \to \mathbb{Z}_p^\times$ as the character such that if $\sigma \in G_K$ and $\zeta \in \overline{K}$ with $\zeta^{p^m} = 1$ for some $m$, then $\sigma(\zeta) = \zeta^{\varepsilon_p(\sigma) \bmod p^m}$.It's important to mention that one can define $\varepsilon_p$ as the unique Galois representation such that if $\sigma \in G_K$ then $\sigma(\zeta_{p^m}) = \zeta_{p^m}^{\varepsilon_p(\sigma) \bmod p^m}$.

Now we want to study what happens if $K$ is a finite extension of $\mathbb{Q}_l$ and $L$ is a finite extension of $\mathbb{Q}_p$.

## 2.1 $p \neq l$: Groethendieck's monodromy theorem

Let $L$ a finite extension of $\mathbb{Q}_p$, and $K$ a finite extension of $\mathbb{Q}_l$ with $p \neq l$ .

**Definition 2.3.** A **representation** of $W_K$ over $L$ is a finite dimensional representation of $W_K$ over $L$ which is continuous if $L$ has the discrete topology (i.e. a representation with open kernel). A **Weil–Deligne representation** of $W_K$ on a finite-dimensional $L$-vector space $V$ is a pair $(r, N)$ consisting of a representation $r : W_K \to GL(V)$, and an endomorphism $N \in End(V)$ such that for all $\sigma \in W_K$ one has

$$r(\sigma)Nr(\sigma)^{-1} = \left(\frac{1}{l}\right)^{v_K(\sigma)}N,$$

where $v_K : W_K \to \mathbb{Z}$ is determined by $\sigma|_{K^{ur}} = Frob_K^{v_K(\sigma)}$.

*Remark* 2.4. $N$ is necessarily nilpotent.

*Remark* 2.5. If we denote by $W'_K = W_K \rtimes \mathbb{G}_a$ the Weil-Deligne group, then Weil-Deligne representation of $W_K$ are just representation of $W'_K$. Weil-Deligne representation are a important objects that compare in the local Langlands conjecture.

The reason why we defined what is a Weil-Deligne representation is the following theorem, that gives us a nice and more concrete way to look at Galois representation,

**Theorem 2.6** (Groethendieck's monodromy theorem)**.** *Let $V$ a finite dimensional $L$-vector space. We fix $\varphi \in W_K$ and $(\zeta_m)_{(m,l)=1}$ a compatible system of $m$-th roots of unity. If $\rho : G_K \to GL(V)$ is a continuous representation, then there is a finite extension $K'/K$ and a unique nilpotent endomorphism $N \in End(V)$ such that for every $\sigma \in I_{K'}$*

$$\rho(\sigma) = exp(Nt_{\zeta,p}(\sigma)).$$

*Moreover, we have an equivalence of categories from the category of continuous representation of $G_K$ on finite-dimensional $L$-vector space, to the category of bounded Weil-Deligne representations on finite dimensional $L$-vector space, that sends $\rho \to (V, r, N)$ with $r(\tau) := \rho(\tau)exp(-t_{\zeta,p}(\varphi^{-v_K(\tau)}\tau)N)$.*

The reason why Weil-Deligne representation are easier than Galois representations is because Weil-Deligne do not depend on the topology on $L$.

## 2.2 $p = l$: p-adic Hodge theory

Galois representation for $p = l$ are much more complicated and there is no simple analogue to Grothendieck monodromy theorem. The study of representation $G_K \to GL_n(\overline{\mathbb{Q}}_p)$ is part of $p$-adic Hodge theory and it has initially been developed around the '80s.

The idea is that we would like to study the representation "coming from geometry": let $X$ being a smooth projective variety over $K$. We have an analogue of the classical Hodge decomposition over $\mathbb{C}$, given by

$$H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p)_{\mathbb{C}_p} \cong \bigoplus_{i+j=n} H^i(X_K, \Omega^j_X) \otimes \mathbb{C}_p(-j),$$

where $\mathbb{C}_p(-j)$ is the Tate twist given by $\mathbb{C}_p \otimes \varepsilon_p^{-j}$. Moreover, this isomorphism is $G_K$ equivariant. Now we need the following theorem:

**Theorem 2.7** (Tate)**.** *For $i \neq 0$,*

$$H^0(G_K, \mathbb{C}_p(i)) = 0.$$

Now we consider the $G_K$ representation $B_{HT}$, given by $\mathbb{C}_p[t, t^{-1}]$ with the action of $G_K$ on $\mathbb{C}_p t^j$ given by $\varepsilon_p^j$. We get that

$$\left(H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{HT}\right)^{G_K} \cong \Big(\bigoplus_{i+j=n, m\in\mathbb{Z}} H^i(X_K, \Omega^j_K) \otimes \mathbb{C}_p(m-j)\Big)^{G_K} \cong \bigoplus_{i+j=n} H^i(X_K, \Omega^j_X).$$

Therefore, the dimension over $K$ of $\left(H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{HT}\right)^{G_K}$ is the same as the dimension of $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p)$. This leads us to the following definition:

**Definition 2.8.** A Galois representation $V$ of $G_K$ is called **Hodge-Tate** if

$$dim_K(V \otimes B_{HT})^{G_K} \cong dim_{\mathbb{Q}_p}(V).$$

One can slightly modify this definition to get different kind of "geometric" representations:

$$\{\textbf{Crystalline}\} \subset \{\textbf{Semi-stable}\} \subset \{\textbf{De Rham}\} \subset \{\textbf{Hodge-Tate}\};$$

we don't have that time to define all of those, but the following theorem should give and idea of how those objects should look like:

**Theorem 2.9.** *If $X$ is a smooth projective variety over $K$, then $H^i_{et}(X_{\overline{K}}, \overline{\mathbb{Q}}_p)$ is a De Rham representation. If $X$ has good (resp. semistable) reduction then $H^i_{et}(X_{\overline{K}}, \overline{\mathbb{Q}}_p)$ is crystalline (resp. semistable).*

The closest analogue to Grothendieck monodromy theorem in this setting, is the following:

**Theorem 2.10** ($p$-adic monodromy theorem)**.** *A representation is de Rham if and only if it is potentially semistable, where potentially semistable means that it's semistable after restricting to the absolute Galois group of a finite extension of $K$.*

Just to finish this section, we define what are Hodge-Tate weights, that are going to be very important in the later talks:

**Definition 2.11.** If $\rho : G_K \to GL(V)$ is a Hodge-Tate representation, then **the Hodge-Tate weights** of $V$ are the $i$ for which $(V \otimes_{\mathbb{Q}_p} \mathbb{C}(i))^{G_K}$ are non-zero and the multiplicity of the weight $i$ is the $K$-dimension of this latter space.

# 3 Galois representation over number fields

In this last section, we will assume $K$ to be a number field, $L$ to be an algebraic extension of $\mathbb{Q}_p$, and $v$ is going to be a place of $K$. Let $K'/K$ be a Galois extension. Since $Gal(K'/K)$ acts on the set of places over $v$, for every place $w$ of $K'$ over $v$, we can define the group

$$Gal(K'/K)_w := \{\sigma \in Gal(K'/K) \mid w = \sigma w\},$$

and there is a natural isomorphism $Gal(K'/K)_w \cong Gal(K'_w/K_v)$ that gives a canonical immersion of $Gal(K'_w/K_v) \hookrightarrow Gal(K'/K)$.

**Definition 3.1.** A Galois representation $\rho : G_K \to GL_n(L)$ is called **geometric** if it is unramified outside of a finite set of places and it's De Rham at each place over $p$. Here by unramified we mean trivial on the inertia.

Again, these geometric representation should be representation coming from étale cohomology. This is the famous Fontaine-Mazur conjecture:

**Conjecture:** Let $\rho : G_K \to GL_n(\overline{\mathbb{Q}_p})$ be a geometric Galois representation. Then $\rho$ is (the extension of scalar of) a subquotient of a representation of the form $H^i_{et}(X_{\overline{K}}, \mathbb{Q}_p)^{ss}(j)$ where $j$ denotes the Tate twist.