Talk 4. Unramified L-packets

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22/05/2023

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Notation

For the entire talk, we fix G/F to be a quasi-split connected reductive group defined over a non-archimedean local field F of characteristic 0. Later in the talk we will require G to be unramified. We denote by $\mathcal{O}_F$ the ring of integer of F and by $\mathfrak{f}$ the residue field of F.

1 Introduction

The goal of this talk is to describe the L-packets of "unramified L-parameters". There are probably many reasons one might want to do this, but here are two:

1. Unramified L-parameters are in some sense the simplest type of L-parameter, so describing their L-packets is a good starting point for wanting to understand the Local Langlands map LL
2. Automorphic representations factor into local representations which turn out to be unramified at almost all places. So understanding these will be important for understanding automorphic forms.

The word unramified will come from these being analogous to unramified Galois representations, as will be clear later on.

2 The basic objects

2.1 Unramified groups

The objects we will describe can only be defined on so-called "unramified groups" which we now introduce. A simple motivation for these groups can be given as follows: in local class field theory we know that the inertia subgroup, I_F corresponds to $\mathcal{O}_F^\times = \mathbb{G}_m(\mathcal{O}_F)$ hence to get a correspondence in general we want to make sense of $G(\mathcal{O}_F)$ for more general groups G. This is what unramified groups accomplish.

Definition 2.1. A compact subgroup K of $G(F)$ is called **hyperspecial** if there exists a smooth affine group scheme $\mathcal{G}$ over $\mathcal{O}_F$ such that the following conditions hold:

1. $\mathcal{G}(\mathcal{O}_F) = K$;
2. $\mathcal{G}_F \cong G$;
3. $\mathcal{G}_{\mathfrak{f}}$ is a connected reductive group.

Remark 2.2. For the ones who know Bruhat-Tits theory. One can prove that if G is semisimple and unramified, then a compact subgroup is hyperspecial if and only if it's the stabilizer of a hyperspecial vertex[1] in the Bruhat-Tits building. In our setting, we have that G is unramified if, and only if, $\mathcal{B}(G)$ has a hyperspecial point.

Example 2.3.

1. The main example of a hyperspecial maximal compact subgroup of $G(F)$, is $G(\mathcal{O}_F)$.
2. Let $G = SL_2/\mathbb{Q}_p$. Then one can show, via Brhuat-Tits theory, that there are 2 conjugacy class of maximal compact hyperspecial subgroup, with representative $K_1 = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ and $K_2 = \begin{pmatrix} \mathbb{Z}_p & p\mathbb{Z}_p \\ p^{-1}\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$

We now describe for what class of groups hyperspecial subgroups exist:

Definition 2.4. An algebraic reductive group is called **unramified** if it's quasi-split, and it splits over an unramified extension.

Example 2.5. $U(1)$ associated to some unramified quadratic field extension E/F is quasi-split but not split, and splits over E which is unramified by assumption. Hence this gives a non-trivial example of a unramified group.

These turn out to all the groups admitting hyperspecial compact subgroups:

Theorem 2.6. *G is unramified iff there exists a hyperspecial compact subgroup in $G(F)$.*

For the remainder of the talk: fix G a unramified reductive group over F, and $K \leq G(F)$ a hyperspecial subgroup.

2.2 Unramified representations

Definition 2.7. Let K be a hyperspecial maximal compact subgroup of $G(F)$. A representation (π, V) of $G(F)$ is called **spherical** with respect to K (or K-spherical) if $V^K \neq 0$. (π, V) is called **unramified** if it is spherical with respect to some hyperspecial maximal compact subgroup.

A simple example of these would be characters of split tori T trivial on $T(\mathcal{O}_F)$. For $T = \mathbb{G}_m$ one sees that they correspond to unramified characters of the Galois group via the local reciprocity map, which gives some motivation for the term unramified.

2.3 Unramified L-parameters

Since we have assumed our group is unramfied, we take L^G to be $\widehat{G}(\mathbb{C}) \rtimes Fr^{\mathbb{Z}}$, where Fr is some choice of geometric Frobenius element. This is needed to define unramified L-parameters:

Definition 2.8. We say that an L-parameter $\varphi : W_F' \to^L G$ is unramified if it's trivial on I_F and $SL_2(\mathbb{C})$.

[1] A hyperspecial vertex is a vertex that is special and keeps being special after any base change to an unramified extension

Giving an unramified L-parameter then simply corresponds to picking a semisimple element $\phi(Fr)$ in the coset $\widehat{G}(\mathbb{C}) \rtimes Fr$.

One can simplify this further, by a "frobenius twisted" version of the fact that semisimple elements in connected groups can be conjugated into any maximal torus:

$$\{\text{unramified L-parameters}\}/\widehat{G}(\mathbb{C})\text{-conj} \longleftrightarrow (\widehat{T}(\mathbb{C}) \rtimes Fr)/_F N(\mathbb{C}) \longleftrightarrow (\widehat{T} \rtimes Fr/_F N)(\mathbb{C})$$

Here $_F N$ is defined as the constant $\mathbb{C}$-group scheme with $\mathbb{C}$-points:

$$_F N(\mathbb{C}) := \{n \in N_{\widehat{G}}(\widehat{T})(\mathbb{C}) : [n] \in W(\widehat{G}, \widehat{T})(\mathbb{C}) \text{ is stable under frobenius}\}$$

It acts on $\widehat{T} \rtimes Fr$ via conjugation in $^L G$. This final parametrization of unramified L-parameters will be used to relate them to unramified representations via the Satake isomorphism. The here makes sense as a GIT quotient since $\widehat{T} \rtimes Fr$ is affine.

3 The Satake isomorphism

To state the Satake isomorphism we need to know some things about Hecke algebras:

3.1 Hecke Algebras

The Hecke algebras are "smooth" analogues of group rings, in the sense that (smooth) modules over them will correspond to smooth $G(F)$ representations. For details on this see [Hah22, Chapter 5]

Definition 3.1. The Hecke algebra of a reductive group G is defined as the (non-unital!) algebra $\mathcal{H}(G(F)) := (C_c^\infty(G(F)), *)$, where $*$ is given by convolution of functions.

Definition 3.2. A smooth Hecke module is a nondegenerate $\mathcal{H}(G(F))$-module, i.e. a module V such that $\mathcal{H}(G(F))V = V$

Now any smooth $G(F)$-representation (V, π) gives rise to a Hecke module by the following action:

$$\pi(f)v := \int_{G(F)} f(g)\pi(g)v d\mu$$

This is analogous to how one in the finite groups setting gives a $\mathbb{Z}[G]$-module structure to a G-representation.

We also have unramified versions of the above:

Definition 3.3. The **unramified Hecke algebra** of $G(F)$, denoted $\mathcal{H}(G(F), K)$, consists of the K-bi-invariant functions in $\mathcal{H}(G(F))$.

By restricting the action defined above, we see that for any unramified representation (V, π), V^K gets the structure of a $\mathcal{H}(G(F), K)$-module. We now have the following key results:

Theorem 3.4. *The above construction gives us an equivalence of categories:*

$$\{\textit{smooth } G(F)\textit{-representations}\} \longleftrightarrow \{\textit{smooth } \mathcal{H}(G(F), K)\textit{-modules}\}$$

As well as a bijection:

$$\{\textit{irreducible smooth } K\textit{-unramified } G(F)\textit{-representations}\} \longleftrightarrow \{\textit{simple smooth } \mathcal{H}(G(F), K)\textit{-modules}\}$$

$$V \mapsto V^K$$

In particular, two unramified irreducible representations (π_0, V_0), (π_1, V_1) are isomorphic as smooth representations iff $V_0^K \cong V_1^K$ as $\mathcal{H}(G(F), K)$-modules.

3.2 Satake isomorphism

The Satake isomorphism will describe the unramified Hecke algebra, and by the above it will therefore also help us to describe the irreducible unramified representations. Let us fix the notation $\mathbb{C}[X]$, to denote the global sections of a $\mathbb{C}$-scheme X. This lets us state the following:

Theorem 3.5 (the Satake isomorphism)**.** *We have an isomorphism:*

$$\mathcal{H}(G(F),K) \xrightarrow{\sim} \mathbb{C}[\widehat{T} \rtimes Fr]^{_F N}$$

We get the following immediate corollary:

Corollary 3.6. $\mathcal{H}(G(F),K)$ *is commutative, in particular simple modules over it are 1-dimensional.*

Using this corollary, and the Satake isomorphism gives us the following parametrization:

$$\begin{aligned}\{\text{irreducible smooth K-unramified } G(F)\text{-representations}\}/\sim &\longleftrightarrow Hom(\mathcal{H}(G(F),K),\mathbb{C}) \\ &\longleftrightarrow Hom(\mathbb{C}[\widehat{T} \rtimes Fr]^{_F N},\mathbb{C}) \\ &\longleftrightarrow (\widehat{T} \rtimes Fr/_F N)(\mathbb{C})\end{aligned}$$

4 Principal series

4.1 Spherical representations

From now on, we fix a maximal hyperspecial compact subgroup K of an unramified group $G(F)$. What we want to do now, is giving an explicit construction of the K-spherical irreducible representations. Before doing that, we want to give the definition of unramified character and we want to recall the statement of the Iwashawa decomposition.

Definition 4.1. We have a valuation map $\omega_G : G(F) \to X_*(G)$ defined as the map such that

$$\langle \omega_G(g), \varphi \rangle = val_F(\varphi(g)).$$

We denote by $G(F)^1$ the kernel of the valuation map. A character χ of $G(F)$ is called **unramified** if it vanishes over $G(F)^1$.

Example 4.2.

1. Let G being semisimple. Then, $G(F)^1 = G(F)$. So the only unramified character of a semisimple group is the trivial character.

2. If $G = \mathbb{G}_m$, then the map ω_G is just the valuation map. We can check this: let $x \in F^\times$, and let $\varphi = n \in X^*(\mathbb{G}_m) \cong \mathbb{Z}$. Then we have

$$n \cdot \omega_G(x) = \langle \omega_G(x), \varphi \rangle = val(\varphi(x)) = val(x^n) = n \cdot val(\varphi(x)),$$

so we get that $\omega_G(x) = val(x)$. Then $\mathbb{G}_m(F)^1 = \mathcal{O}_F^\times$. We can generalize this and we get, that for a general split torus T, we have $T(F)^1 = T(\mathcal{O}_F)$.

Remark 4.3. The unramified characters of a torus $T(F)$ are exactly the irreducible unramified representation of $T(F)$.

Now we give the statement of the Iwasawa deomposition. This is a standard result that can be proven with Bruhat-Tits theory.

Theorem 4.4 (Iwasawa decomposition)**.** *There exists a maximal torus T and a Borel subgroup $B = TU$ with unipotent radical U, such that $G(F) = B(F)K$.*

From now on we fix B and T as the ones in the Iwasawa decompositon. For the construction of the K-spherical irreducible representations, we are going to follow [C+79]. We first give an outline of the construction. The point is that every K-spherical irreducible representation, is associated to something called a K-spherical function, and every K-spherical function comes from some unramified character of the torus of $T(F)$. So, first we want to define what are these K-spherical functions:

Definition 4.5. A (zonal) **spherical function** on $G(F)$ with respect to K, (or just a K-spherical function) is a function $\Gamma : G(F) \to \mathbb{C}$, bi-invariant under K, such that $\Gamma(1) = 1$ and

$$\Gamma(g_1)\Gamma(g_2) = \int_K \Gamma(g_1 k g_2) dk \text{ for } g_1, g_2 \in G(F). \tag{1}$$

The question now is: how can we get a spherical function from an unramified character of $T(F)$? Consider an unramified character χ of $T(F)$ and consider the function $\phi_{K,\chi}$ given by

$$\phi_{K,\chi}(tuk) = \delta(t)^{1/2}\chi(t) \text{ for } t \in T, u \in U, k \in K.$$

Here $\delta(t)$ is what is called the *modulus function*[2]. Set

$$\Gamma_\chi^K(g) = \int_K \phi_{K,\chi}(kg) dk \text{ for } g \in G.$$

Theorem 4.6. *1. Γ_χ^K is a K-spherical function.*

2. Every K-spherical function on $G(F)$ with respect to K is of the form Γ_χ^K for some unramified character χ.

3. Two different unramified character χ, χ' give the same K-spherical function if, and only if, there exists an element $w \in W$ such that $\chi = w \cdot \chi'$.

Proof. The fact that Γ_χ^K is K-bi-invariant and $\Gamma(1) = 1$ is easy. For the rest of the proof we recommend [C$^+$79, Theorem 4.2]. □

Given a spherical function, we can know construct a spherical representation associated to it. Let Γ be a spherical function on $G(F)$ (w.r.t. K) and denote by V_Γ the space of functions f on $G(F)$ of the form

$$f(g) = \sum_{i=1}^{n} c_i \Gamma(g g_i)$$

for some $c_1, \ldots, c_n \in \mathbb{C}$ and $g_1, \ldots, g_n \in G$. We let G act on V_Γ by right translations, namely

$$(\pi_\Gamma(g) \cdot f)(g') = f(g'g) \text{ for } g, g' \in G, f \in V_\Gamma. \tag{2}$$

Theorem 4.7. *1. The representation (π_Γ, V_Γ) is irreducible spherical (w.r.t. K) and the elements of V_Γ invariant under K are the constant multiples of Γ.*

2. Let (π, V) be any K-spherical irreducible representation of $G(F)$. There exists a unique spherical function Γ such that (π, V) is isomorphic to (π_Γ, V_Γ).

Proof. 1. V_Γ is K-spherical since, $\Gamma \in V_\Gamma^K$. The irreducibility, one can prove that each subrepresentation must contain Γ using the third property of spherical functions 1. Now we divide the proof of the theorem in steps:

We will just give a sketch of the proof. For a more detailed proof, we recomend [C$^+$79, Theorem 4.3].

(a) First we prove that V^K is a simple $\mathcal{H}(G(F), K)$ module.

(b) Then, since we know from the Satake isomorphism that $\mathcal{H}(G(F), K)$ is commutative, V^K must be 1-dimensional.

(c) We consider the controgradient representation $\widetilde{V}$ of V, and we consider $\widetilde{V}^K$ that must be 1-dimensional since it's isomorphic to the dual of V^K. The we chose $v \in V$, and $\tilde{v} \in \widetilde{V}$ such that $\langle \tilde{v}, v \rangle = 1$. Then we can define the function

$$\Gamma(g) = \langle \tilde{v}, \pi(g) \cdot v \rangle \text{ for } g \in G(F).$$

(d) We prove that $(\pi, V) \cong (\pi_\Gamma, V_\Gamma)$.

□

[2]The modulus function is defined as $\delta(t) := |detAd_{\mathfrak{u}}(t)|_F$ where $\mathfrak{u}$ is the Lie algebra of U. For us is not really relevant, it's just a correction factor that we need for having nice properties.

4.2 Unramified principal series

Now, we want some representation that, in some sense, contains all the possible unramified representation associated to some unramified character of some torus. This will be helpfull to state the local Langlands conjecture for the unramified case.

Let χ be an unramified character of $T(F)$. We define the unramified principal series $PS(\chi)$[3] of $G(F)$ to be the following representation: $PS(\chi)$ consists of the locally constant function $f : G(F) \to \mathbb{C}$ such that

$$f(tug) = \delta(t)^{1/2}\chi(t)f(g), \text{ for } t \in T, u \in U, g \in G.$$

The group acts on this space via right translation, namely

$$g \cdot f(g') = f(g'g) \text{ for } f \in PS(\chi),\ g, g' \in G(F).$$

One can notice that this is just the induction of χ, with an adjustment given by $\delta^{1/2}$.

Theorem 4.8. *1. Assume that $PS(\chi)$ is irreducible. Then it is isomorphic to $(\pi_{\Gamma_\chi^K}, V_{\Gamma_\chi^K})$ via the map that sends $f \to f^\sharp$ defined as $f^\sharp(g) = \int_K f(kg)dk$.*

2. In general, let $0 = V_0 \subset V_1 \subset \cdots \subset V_r = PS(\chi)$ be a Jordan-Holder series of the $\mathcal{H}(G(F))$-module $PS(\chi)$. There exists a unique index j such that V_j/V_{j-1} is K-spherical. In that case, this quotient is isomorphic to $(\pi_{\Gamma_\chi^K}, V_{\Gamma_\chi^K})$.

So, in some sense we can think of the unramified principal series associated to χ, as a representation that "contains" all the unramified representation associated to χ.

5 The correspondence for tori

In this section we are going to state some results about the correspondence for tori. A more complete reference for this is [Bor79, Section 9]. We consider T an unramified torus over F.

Theorem 5.1. *There is a bijection $\Pi(T) \longleftrightarrow \Phi(T)$.*

Proof. [Lan68, Theorem 1]. □

Moreover, T has only one maximal hyperspecial compact subgroup up to conjugaction, that is $T(F)^1$. Therefore, if $\varphi \in \Phi_{ur}(T)$ is an unramified L-parameter, this is associated to some semisimple element in $\widehat{T}(\mathbb{C}) \rtimes Fr$, and this corresponds, thanks to a corollary of the Satake isomorphism, to an unramified character $\chi \in \Pi_{ur}(T)$. This gives us a bijection between the set $\Pi_{ur}(T)$ of unramified character and $\Phi_{ur}(T)$ of unramified L-parameter.

Remark 5.2. If $G = GL_2$, then it's still true that there is only 1 conjugacy class of hyperspecial maximal compact subgroup. So, the same argument as before, proves that we have a bijection between $\Pi_{ur}(G)$ and $\Phi_{ur}(G)$.

6 The correspondence in the unramified case

References for this section, are [Bor79, Section 10.5] and [Mis12]. In this section T is any maximal torus of G. Let $\varphi \in \Phi_{ur}(G)$. Being unramified means that φ vanishes on SL_2 and I_F. It factors through W_F/I_F that is abelian, so it's image is abelian and contained in a maximal torus of ${}^LG$. Since φ is defined up to conjugation, we can assume that the image of φ is contained in ${}^LT$.

So, we can see φ as an element in $\Phi(T)_{ur}$. Using the correspondence for tori, we get an unramified character $\chi_\varphi =: \chi$ of $T(F)$ associated to φ. Now we can construct the unramified principal series $PS(\chi)$, and we want to describe the packet Π_φ via $PS(\chi)$.

THE CONJECTURE: We require Π_φ to consist of the factors of the Jordan-Holder series of $PS(\chi)$ that are unramified.

[3][C$^+$79] uses $I(\chi)$ instead of $PS(\chi)$ as notation

Example 6.1. This example is actually incomplete, but we present what we have got so far.

Let $G = SL_2/\mathbb{Q}_p$, and consider the unramified L-parameter φ that sends $Fr \to \tau = \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The point of $\widehat{T}(\mathbb{C})$ corresponds to point in $X_*(\widehat{T}) \otimes \mathbb{C}^\times$ by evaluating. So, τ corresponds to the element $(1, 0) \otimes x$. From this, we get that τ corresponds to the unramified character $\chi : T(F) \to \mathbb{C}^\times$ that sends $t = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \to x^{val(a)}$.

Now, first we notice that if $x = 1$, then the centralizer of the image of φ, is given by $G(F)$, and it's equal to $T(F)$ if $x > 1$.

If $\tau = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ then, the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is contained in the centralizer of the image of φ. Therefore, one can compute that $\pi_0(\overline{S_\varphi})$ non-trivial, and it is in fact isomorphic to $\mathbb{Z}/2\mathbb{Z}$. So, we expect to find a non trivial L-packet if $x = -1$.

Now one can try to understand $PS(\chi)$. We can prove (using [C$^+$79, Theorem 3.5]) that $End(PS(\chi))$ is actually 2 dimensional if $x = -1$, and this implies that $PS(\chi)$, in this case, is not irreducible. Actually, one can prove, using the same theorem, that $PS(\chi)$ is irreducible exactly when we expect it to be.

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