Talk 8, Moy-Prassad filtration

Giulio Ricci

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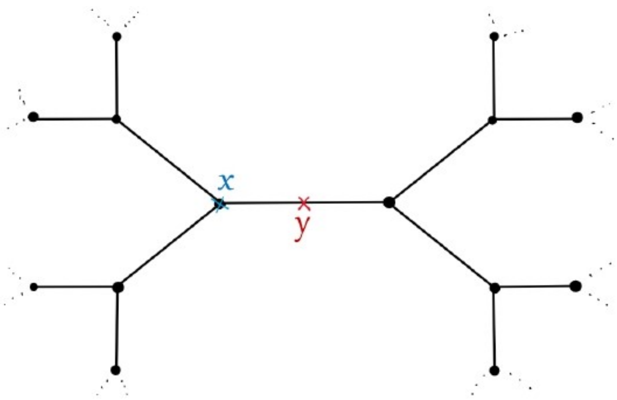
The goal of this talk is defining the Moy-Prassad filtration for a connected reductive quasi-split algebraic group, and try to understand this is an interesting tool for the studying of the representation of the group. We are going to start with an example from last semester:

Example 0.1. Consider the group $G = SL_2/\mathbb{Q}_p$, and consider the associated building:



With x the point associated with the subgroup $G_x := \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$, and y the point associated to the subgroup $G_y := \begin{pmatrix} \mathbb{Z}_p & p\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$. Last semester, we described the vertices of the building in terms of graded period lattice chains.

Given a point in the building $w = (\mathcal{L}_\alpha, c+r) \in \mathcal{B}(G)$, we defined the Moy-Prassad filtration subgroups via

$$G_{w,s} := \{g \in SL(V) \mid (g-1)\mathcal{L}_{\alpha,r} \subset \mathcal{L}_{\alpha,r+s}\}$$

and we got 2 filtrations:

$$G_{x,0} = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \supset G_{x,1} = \begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix} \supset G_{x,2} = \begin{pmatrix} 1+p^2\mathbb{Z}_p & p^2\mathbb{Z}_p \\ p^2\mathbb{Z}_p & 1+p^2\mathbb{Z}_p \end{pmatrix} \cdots$$

and

$$G_y = \begin{pmatrix} \mathbb{Z}_p & p\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \supset G_{y,\frac{1}{2}} = \begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ \mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix} \supset G_{y,1} = \begin{pmatrix} 1+p\mathbb{Z}_p & p^2\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix} \supset G_{y,\frac{3}{2}} := \begin{pmatrix} 1+p^2\mathbb{Z}_p & p^2\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p^2\mathbb{Z}_p \end{pmatrix} \cdots$$

These are the Moy-Prassad filtrations for G_x and G_y respectively. The goal of this talk is to generalize this construction to a general reductive group.

Notice that we can already see some of the good properties that we want the Moy-Prassad filtration to satisfy:

1. $G_{x,r}$ is compact, open and normal in $G_{x,0}$.

2. The filtration is separated[1].

3. $G_{x,0}/G_{x,0+}$[2] is the group of the $\mathbb{F}_p$ points of an $\mathbb{F}_p$-reductive group $\mathbb{G}_x$.

[1] A filtration is called separated if the intersection of the entire filtration is 0. It's called exhaustive if the union is the entire group.

[2] We denote $G_{x,0+} = \bigcup_{s>0} G_{x,s}$.

4. $G_{x,r}/G_{x,r+}$ is an $\mathbb{F}_p$ vector space.

From now on, we will use the following usual notation: k is field with a discrete valuation $\omega : k^\times \to \mathbb{R}$, $\mathfrak{o}$ it's its ring of integers with maximal ideal $\mathfrak{m}$ and residue field $\mathfrak{f}$. For this talk we will assume that $\mathfrak{f}$ is perfect and $\omega(k^\times) = \mathbb{Z}$. We denote K the maximal unramified extension of k, $\mathcal{O}$ its ring of integers, and $\Gamma = Gal(K/k)$. Finally, G will be a reductive group over k.

From last semester, we already know that fixing a point in $x \in \mathcal{B}(G,k)$, we get a descending filtration of the root groups U_α given by the open compact subgroups

$$U_{\alpha,x,r} := \{u \in U_\alpha \mid \varphi_{x,\alpha}(u) \geq r\}.$$

What we need now is a filtration of the centralizer of a maximal split torus S.

Filtration for $Z_G(S)$

Main reference: Chapter 7 of [2]

We start with a connected reductive group G over k, a maximal split torus S and its centralizer $Z = Z_G(S)$. Recall that Z is a maximal torus if G is quasi-split.

More general, we want to define a filtration for any torus T/k, and we want this filtration to be "good". (Separable, commutator friendly,..)

We would like to define our filtration in this way:

1. $T(K)_0 = T(K)^0$ the Iwahori subgroup.

2. If T is induced[3], and $r \geq 0$, then

$$T(k)_r := \{t \in T(k)_0 \mid \forall \chi \in X^*(T) : \omega(\chi(t) - 1) \geq r\}.$$

The filtration defined via (1) and (2) is called **standard filtration**, and it has some really good properties if T has at least induced wild ramification.

Example 0.2. If $G = GL_n$ over $\mathbb{Q}_p$, and T is the torus consisting of diagonal matrices, then $T(F)^0$ consists of all diagonal matrices with entries in $\mathcal{O}^\times$ and $T(F)_r$ consists of all diagonal matrices with entries in $1 + \varpi^{\lceil r \rceil}\mathcal{O}$.

If T has induced wild ramification, the standard filtration induces a filtration on the Lie algebra

$$\mathfrak{t}(k)_r := \{X \in \mathfrak{t}(k) \mid \omega(d_\chi(X)) \geq r \text{ for every } \chi \in X^*(T)\}.$$

Moreover we have a functorial isomorphism, called the Moy-Prassad isomorphism

$$T(k)_r/T(k)_s \to \mathfrak{t}(k)_r/\mathfrak{t}(k)_s.$$

More in general, one can construct a filtration called **minimal congruent filtration**, that has the properties that we want and that coincides with the standard filtration when T has induced wild ramification.

But what does it mean to be good?

Definition 0.3. Consider $\{T(k)^*_r\}_{r\in\mathbb{R}}$ a filtration of $T(k)^0$. Then the filtration is called:

- **Functorial** if for every two tori $T_1/k, T_2/k$ and every map $f : T_1 \to T_2$ it satisyies $f(T_1(k)^*_r) \subset T_2(k)_r*$;

- **Admissible** if it coincides with the standard filtration is T is induced, $T(k)^*_0 = T(k)^0$ and $T(k)^*_{0+}$ is the group of elements having unipotent image in $\mathscr{T}^0(\bar{\mathfrak{f}})$.

[3] A k-torus T is called induced if the lattice $X^*(T)$, or equivalently $X_*(T)$, has a $\mathbb{Z}$-basis that is invariant under the Galois group of the splitting extension of T. e.g. G simply connected or adjoint, then a maximal torus is induced. A torus as induced wild ramification if there exists a tamely ramified extension l of k, such that T_l is induced.

When we say "good", we mean that these two properties hold, that we have a Moy-Prassad isomorphism, and that the subgroups of the filtration are schematic and connected.

The good thing for us, is that, the result that we are about to present, just requires the existence of "good" filtration. Therefore, from now on, we will assume that we have a "good" filtration $Z(k)_r \subset Z(k)$ in G.

Concave function and associated parahoric subgroup

Main references: Chapter 7.3, Chapter 14.2 [2]

Consider Φ a root suystem for G, and $\widehat{\Phi} = \Phi \cup \{0\}$.

Definition 0.4. A function $f : \widehat{\Phi} \to \tilde{\mathbb{R}}$[4] is called **concave** if $f(a+b) \leq f(a) + f(b)$ for every $a, b \in \widehat{\Phi}$.

Definition 0.5. Let $x \in \mathcal{A}(S)$, and f concave. Then:

1. $U_{a,x,f} = U_{a,x,f(a)} U_{2a,x,f(2a)}$ for $a \in \Phi$.
2. We set $G(k)^{\sharp}_{x,f}$ to be the subgroup generated by the $U_{a,x,f}$ with $a \in \Phi$.
3. $\mathcal{P}_{x,f} := G(k)_{x,f} := G(k)^{\sharp}_{x,f} \cdot Z(k)_{f(0)}$.

Remark 0.6. Notice that the definition of $\mathcal{P}_{x,f}$ depends on the choice of the filtration of $Z(G)$.

Remark 0.7. Even if $\mathcal{P}_{x,0}$ depends only on the facet in which x is contained, this doesn't hold for $\mathcal{P}_{x,f}$.

Theorem 0.8. *Let $f : \widehat{\Phi} \to \tilde{\mathbb{R}} \setminus \{\infty\}$ be a concave function. Then the subgroup $G(k)_{x,f}$ is schematic and connected. We will write $\mathscr{G}_{x,f}$ for its corresponding smooth model with connected fibers. Moreover, the product morphism*

$$\prod_{\alpha \in \Phi^{-,nd}} \mathscr{U}_{\alpha,x,f} \times \mathscr{Z}_{f(0)} \times \prod_{\alpha \in \Phi^{+,nd}} \mathscr{U}_{\alpha,x,f} \to \mathscr{G}_{x,f}$$ [5]

is an open immersion, and the product over $\Phi^{-,nd}$ and $\Phi^{+,nd}$ can be taken arbitrary. Here $\mathscr{U}_{\alpha,x,f}$, $\mathscr{Z}_{f(0)}$ denote the integral model of $U_{\alpha,x,f}$ and $Z(k)_{f(0)}$ respectively.

Proof. The proof of this theorem is complicated and we are not going to do it. Reference: Section 8.5, Theorem 8.5.2, [2]. ☐

Definition 0.9. We define the **Moy–Prasad filtration subgroup** to be $\mathcal{P}_{x,r}$ with $r \in \mathbb{R}$ seen as a constant function (so concave). Notice that, $\mathcal{P}_{x,0} = G(k)^0_x$.

Remark 0.10. Pick $x \in A(T_K)$ in $\mathcal{B}(G_K)$, we can get define analogously the Moy-Prassad filtration $P_{x,r}$ of G_K. In this case, $\mathcal{P}_{x,r} = (P_{x,r})^{\Gamma}$.

Remark 0.11. One can show that $\mathcal{P}_{x,r}$ is a bounded open subgroup.

Proposition 0.12. *The filtration $\{\mathcal{P}_{x,r}\}_{r \in \mathbb{R}}$ is a decreasing separated filtration of $G(k)$ by bounded open subgroups with the following properties:*

1. *The group $\mathcal{P}_{x,r}$ depends only on x and r, not on S;*
2. *For any $g \in G(k)$, $g\mathcal{P}_{x,r}g^{-1} = \mathcal{P}_{gx,r}$. In particular, $\mathcal{P}_{x,r}$ is normal in $G(k)_x = Stab_G(x)$.*
3. *If $r > 0$ then the product map*

$$\prod_{\alpha \in \Phi^+} U_{\alpha,x,r} \times Z(k)_r \times \prod_{\alpha \in \Phi^-} U_{\alpha,x,r} \to \mathcal{P}_{x,r}$$

is a bijection and the factor in the product over Φ^+ and Φ^- can be taken in every order.

[4] $\tilde{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$

[5] $\Phi^{nd} = \{a \in \Phi \mid a/2 \notin \Phi\}$.

4. $[\mathcal{P}_{x,r}, \mathcal{P}_{x,s}] \subseteq \mathcal{P}_{x,r+s}$.

Remark 0.13. Consider $x \in \mathcal{B}(G)$, $\mathscr{G}_x^0$ be the parahoric group scheme with connected fibers. Let $\overline{\mathscr{G}}_x^0$ the special fiber of $\mathscr{G}_x^0$ and $\mathbb{G}_x$ the maximal reductive quotient of $\overline{\mathscr{G}}_x^0$. The surjective morphism $\mathcal{P}_{x,0} = \mathscr{G}_x^0(\mathfrak{o}) \to \mathbb{G}_x(\mathfrak{f})$ has kernel $\mathcal{P}_{x,0+}$. Therefore, we can identify $\mathbb{G}_x(\mathfrak{f})$ with $\mathcal{P}_{x,0}/\mathcal{P}_{x,0+}$. More generally, we have an identity of $\mathfrak{f}$-group $\mathbb{G}_x = Cok(\overline{\mathscr{G}}_{x,0} \to \overline{\mathscr{G}}_{x,0+})$ where the last map is induced by the inclusion of $G(k)_{x,0+} \subset G(k)_{x,0}$.

$\mathbb{G}_x$ will be very important for studying the filtration.

Remark 0.14. The quotient $G(k)_{x,r}/G(k)_{x,r+}$ it's a vector space over $\mathfrak{f}$.

Filtration of the Lie Algebra

Main reference: Chapter 14.3 [2]

Let $x \in \mathcal{B}(G)$ and consider for $r \in \mathbb{R}_{\geq 0}$ the group scheme $\mathscr{G}_{x,r}$. We call $\mathfrak{g}_{x,r}$ the Lie algebra of this group scheme. This forms an $\mathfrak{o}$-lattice in $\mathfrak{g}(k)$. The Lie algebra is often easier to study than the group. We have a nice identity given by $\mathfrak{g}_{x,r+n} = \mathfrak{m}^n \cdot \mathfrak{g}_{x,r}$ with $\mathfrak{m}$ the maximal ideal in $\mathfrak{o}$, and the sequence $\{\mathfrak{g}_{x,r}\}$ defines a descending separated filtration of $\mathfrak{g}(k)$.

We can give a more explicit description of $\mathfrak{g}_{x,r}$. Namely, if $\mathfrak{g}_{x,r}(\mathcal{O})$ is the $\mathcal{O}$-lattice in $\mathfrak{g}(K)$ given by the Lie algebra of $\mathscr{G}_{x,r} \times_{\mathfrak{o}} \mathcal{O}$, then $\mathfrak{g}_{x,r} = \mathfrak{g}_{x,r}(K)^{\Gamma}$.

Now, from theorem 0.8, we get a nice description of the $\mathcal{O}$-lattice $\mathfrak{g}_{x,r}(\mathcal{O})$ inside $\mathfrak{g}_{x,r}(K)$. In fact, we know that the map

$$\bigoplus_{\alpha \in \Phi^{+,nd}} \mathfrak{u}_{\alpha,x,r} \oplus \mathfrak{z}_r \oplus \bigoplus_{\alpha \in \Phi^{-,nd}} \mathfrak{u}_{\alpha,x,r} \to \mathfrak{g}_{x,r}(\mathcal{O})$$

is an isomorphism of $\mathcal{O}$-modules, where $\mathfrak{u}_{\alpha,x,r}, \mathfrak{z}_r$ are the Lie algebras of $\mathscr{U}_{\alpha,x,r}$ and $\mathscr{Z}_r$ respectively.

But why did we introduce this tool? Even if it's easier to study, at the moment we don't know how this Lie algebras are linked with the Moy-Prassad filtration subgroups. That's why, the next section it is crucial.

Moy-Prassad isomorphism

Our next goal it to prove the there is a Moy-Prassad isomorphism analogous to the case of the filtration of the torus. For this, we require that the chosen filtration for the torus respects the Moy-Prassad isomorphim.

Now, fix 2 positive real numbers $0 \leq r \leq s$, and consider the following conditions:

1. The maximally split maximal torus of G_K has induced wild ramification.
2. $r = r_0 + n$ with $0 \leq r_0 < 1$ and n an integer. and $s \leq r_0 + 2n$.
3. $0 < r \leq s \leq 2r \leq r + 1$.

Note that these 3 conditions are not mutaly exclusive.

Theorem 0.15. *Assume that one of the last 3 conditions hold.*

1. *There exists an isomorphism of abstract abelian groups*

$$MP_{x,r,s} : G(k)_{x,r}/G(k)_{x,r} \to \mathfrak{g}(k)_{x,r}/\mathfrak{g}(k)_{x,s}$$

compatible with unramified algebraic extension.

2. *If either one of condition 1 and 2 holds, then the isomorphism can be chosen with the following property: For any k-rational automorphism, θ of G, then*

$$d\theta \circ MP_{x,r,s} \circ \theta^{-1} = MP_{\theta(x),r,s}.$$

3. *When both conditions (1) and (2) hold simultaneously, the isomorphism they lead coincide.*

Remark 0.16. Let $x \in \mathcal{B}(G)$. $G(K)_{x,r}$ is normal in $G(K)_x^0$, so we have the conjugation action of $G(K)_x^0$ on $G(K)_{x,r}$. This action extends to an action of $\mathscr{G}_x^0$ on the group scheme $\mathscr{G}_{x,r}$ and hence, on the Lie algebra $\mathfrak{g}_{x,r}$.

This action descends now to a rational action of $\overline{\mathscr{G}_x^0}$ on $\mathfrak{g}_{x,r} \otimes_{\mathfrak{o}} \mathfrak{f}$, that induces an $\mathfrak{f}$-rational action of $\mathbb{G}_x$ on $\mathfrak{g}_{x,r} \otimes_{\mathfrak{o}} \mathfrak{f}/\mathfrak{g}_{x,r+} \otimes_{\mathfrak{o}} \mathfrak{f}$.

Example 0.17. Let's consider the first example that we had in this talk: $G = SL_2/\mathbb{Q}_p$.

We had the filtration of

$$G_{x,0} = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \supset G_{x,1} = \begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix} \supset G_{x,2} = \begin{pmatrix} 1+p^2\mathbb{Z}_p & p^2\mathbb{Z}_p \\ p^2\mathbb{Z}_p & 1+p^2\mathbb{Z}_p \end{pmatrix} \cdots$$

The quotient of $G_{x,0}/G_{x,1} \cong SL_2(\mathbb{F}_p)$.[6] The Lie algebra is $\mathfrak{g}_{x,n} := \begin{pmatrix} p^n a & p^n b \\ p^n c & -p^n a \end{pmatrix}$.

Now, the quotient $\mathfrak{g}_{x,1}/\mathfrak{g}_{x,2} \cong \mathfrak{sl}_2(\mathbb{F}_p)$. Moreover, the action of the quotient $G_{x,0}/G_{x,1}$ is just the adjoint action.

For y we have analogous results. y is not special, and $G_{y,0}/G_{y,\frac{1}{2}} \cong \mathbb{G}_m$.

Stable and semi-stable vectors. The Hilbert-Mumford criterion

Let G be a reductive group acting linearly on a vector space V.

Definition 0.18. A non-zero point $\lambda \in V$ is called semi-stable, if 0 is not contained in the closure of its orbit, and unstable otherwise. λ is called stable if its orbit is closed and its stabilizer is finite.

Remark 0.19. Stable implies semi-stable.

Example 0.20. Consider $G = \mathbb{G}_m/\mathbb{C}$ and λ an action on some finite dimensional vector space V.

We can decompose V into a direct sum $V = \bigoplus_i V_i$ where on the i-th component, the action is given by $\lambda(t) \cdot v = t^i \cdot v$. We call i the weight.

Now, we look at the set of weights of a point x.

- If all the weights are strictly positive, then $lim_{t\to 0}\lambda(t) \cdot x = 0$, so x is in the closure of the orbit, and x is unstable.
- If all the weights are non-negative, with 0 being a weight, then either 0 is the only weight in which case x is stabilized by $\mathbb{C}^*$, or there are some positive weights beside 0, and then, $lim_{t\to 0}\lambda(t) \cdot x$ is the weight-0 component of x, that is not in the orbit of x. In both cases, x is semi-stable but not stable.

The Hilbert-Mumford criterion The Hilbert–Mumford criterion essentially says that the multiplicative group case is the typical situation. Precisely, for a general reductive group G acting linearly on a vector space V, the stability of a point x can be characterized via the study of 1-parameter subgroups of G, which are non-trivial morphisms $\lambda : \mathbb{G}_m \to G$.

- A point x is unstable if and only if there is a 1-parameter subgroup of G for which x admits only positive weights or only negative weights;
- A point x is semi-stable if and only if there is no such 1-parameter subgroup, i.e. for every 1-parameter subgroup there are both non-positive and non-negative weights;
- A point x is strictly semi-stable if and only if there is a 1-parameter subgroup of G for which x admits 0 as a weight, with all the weights being non-negative (or non-positive);

[6] This is always true if we consider a split group and a special point.

- A point x is stable if and only if there is no 1-parameter subgroup of G for which x admits only non-negative weights or only non-positive weights, i.e. for every 1-parameter subgroup there are both positive and negative weights.

The case that we are going to study is the following: Consider the action of $G(K)_{x,0}/G(K)_{x,0+}$, on $G(K)_{x,r}/G(K)_{x,r+} =: V_{x,r}$, and consider the dual of this representation. We call $\widehat{V}_{x,r} := Hom(V_{x,r}, \mathfrak{f})$.

What we are going to do, in the next talk, is seeing what happens if we have a stable vector in $\widehat{V}_{x,r}$, following [3].

Remark 0.21. Exercise:

In our favorite example, $G = SL_2$ we can try to compute stable and semi-stable.

References

[1] Jessica Fintzen: *Representation of p-adic groups.*

[2] Tasho Kaletha, Gopal Prassad: *Bruhat-Tits theory: a new approach.*

[3] Mark Reeder, Jiu-Kang Yu: *Epipelagic representation and invariant theory*